Simplicial Sets

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1 CW Complexes

The objects of study in classical homotopy theory are the *homotopy types*. This is not the same thing as a topological space, or even a CW complex, but "CW complex up to homotopy". CW complexes are spaces that admit a construction in stages, starting with some points, then gluing on intervals via their boundary, then gluing on disks via their boundary, and so on, then taking the union of all finite stages. In stage n the "gluing" of n-disks onto the $(n-1)$-skeleton X_{n-1} can be understood categorically as taking a pushout of X_{n-1} with your family of disks $\coprod_{\lambda\in\Lambda} D^n$ along a family of arbitrary continuous maps $\{f_\lambda : S^n \to X\}_{\lambda\in\Lambda}$ ("attaching maps") and standard inclusions $S^n \hookrightarrow D^n$. We could just have easily defined this using (topological) simplex inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for Δ^n and D^n are convex bodies of the same dimension and so canonically (after picking a basepoint) homeomorphic. So CW complexes are *exactly* the topological spaces that can be obtained from a sequential colimit of pushouts of (coproducts of) the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. In other words, they're spaces obtained by gluing simplices together with the restriction that one may only glue along the boundary, but the flexibility that arbitrary continuous gluings of that boundary are allowed. But combining the "Simplicial Approximation Theorem" with the following lemma allows us to assume a CW complex is obtained from a very, very structured kind of gluing.

Lemma 1.1. *Let X be a topological space and $f, g : S^{n-1} \to X$ two homotopic maps. Then the pushouts (or "amalgamation spaces") $D^n \amalg_f X$ and $D^n \amalg_g X$ are homotopy equivalent.*

Proof. Let $H : S^{n-1}\times I \to X$ be a homotopy. The key idea is that we may use the deformation retraction of the "cylinder" $D^n\times I$ onto its boundary minus the top $(D^n\times\{0\})\cup(S^{n-1}\times I)$ to get a deformation retraction of $(D^n\times I)\amalg_H X$ onto $((D^n\times\{0\})\cup(S^{n-1}\times I))\amalg_H X$. We have a morphism $J : (D^n\times I)\amalg_H X \to ((D^n\times\{0\})\cup(S^{n-1}\times I))\amalg_H X$ induced by the morphism of spans

$$\begin{tikzcd}
D^n \arrow[d, hook] & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, equal] \\
(D^n\times\{0\})\cup(S^{n-1}\times I) & & S^{n-1}\times I \arrow[ll, hook'] \arrow[r, "H"] & X.
\end{tikzcd}$$

And in fact J is surjective, because every point in the extra bit $S^{n-1}\times(0,1]$ is glued onto X by H. But it's actually a split monomorphism as well, because morphism of spans above has a left inverse

$$\begin{tikzcd}
D^n & & S^{n-1} \arrow[ll, hook'] \arrow[r, "f"] & X \arrow[d, equal] \\
(D^n\times\{0\})\cup(S^{n-1}\times I) \arrow[u, two heads] & & S^{n-1}\times I \arrow[ll, hook'] \arrow[r, "H"] \arrow[u, two heads] & X.
\end{tikzcd}$$

This means J is actually a homeomorphism, because it is a surjection with a continuous left inverse. The punchline is that $D^n \amalg_f X$, and by symmetry $D^n \amalg_g X$, are both homeomorphic to deformation retracts of the same space (and hence are homotopy equivalent). $\square$

> Exercise: Reprove Lemma 1.1 in terms of the simplicial inclusions, using the fact that Δ^n deformation retracts onto any of its "horns" Λ^n_i (those spaces formed by removing the ith face from $\partial\Delta^n$).

2 The simplex category, gluing, and presheaves

Simplicial sets are a more "algebraic" or "combinatorial" way of modelling homotopy types. This has the advantage that it transports more easily to algebraic contexts. E.g., the (1-)category of topological abelian groups is not abelian but the (1-)category of simplicial abelian groups is! We saw above through careful analysis of CW complexes that any homotopy type is built up from gluing together simplices along their boundaries. For CW complexes the gluing was fairly geometric, an actual pushout in the category of topological spaces. Simplicial sets take the opposite approach: they are formal gluings of (formal!) simplices. Before we can define simplicial sets we must discuss the (category of) simplices from which they are glued.

Definition 2.1. The simplex category Δ has objects the finite nonempty ordinals $[n] = \{0, 1, \dots, n\}$ and a morphism $[n] \to [m]$ is simply an order preserving function. The augmented simplex category Δ_a is defined in the same way, but the empty ordinal $[-1] = \varnothing$ is included.

Note that Δ is equivalent to the category of all finite totally ordered sets. What does this have to do with actual geometric simplices? The object $[n]$ should be understood as a representation of the geometric n-simplex Δ^n, and its elements $0, \dots, n$ representing the $(n+1)$-vertices of that simplex. As demonstrated by simplicial or singular homology, it's often more convenient to work with simplices that have a chosen order on their vertices (for manageably and consistently tracking orientation); this is why we're looking at ordered finite sets and not just finite sets[1]. The geometric simplex Δ^n is the convex hull of its vertices $e_0, \dots, e_n$, and this means that every function of finite sets $\{e_0, \dots, e_n\} \mapsto \{e_0, \dots, e_m\}$ has a unique extension to an affine transformation $\Delta^n \to \Delta^m$ sending vertices to vertices. Thus Δ could just as truthfully be described as the category of geometric simplices $\Delta^n \subseteq \mathbb{R}^{n+1}$ with morphisms the affine transformations sending vertices to vertices and preserving the standard order on those vertices.

Definition 2.2. Let

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \, : \, x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

be the n-dimensional "geometric simplex". The vertices of Δ^n are the standard basis vectors $e_0, \dots, e_n$ of $\mathbb{R}^{n+1}$ and any point in Δ^n can be uniquely represented as a convex combination $t_0 e_0 + \dots + t_n e_n$ of them. Given an order-preserving map $f : [n] \to [m]$ there is an induced continuous map $\widetilde{f} : \Delta^n \to \Delta^m$ defined by

$$\widetilde{f}\left(\sum_{i=0}^{n} t_i e_i \right) = \sum_{i=0}^{n} t_i e_{f(i)}.$$

Exercise: The assignments $[n] \mapsto \Delta^n$ and $f \mapsto \widetilde{f}$ define a faithful functor $\Delta \to \mathsf{Top}$.

There are two important families of maps within Δ, the coface and codegeneracy maps.

Definition 2.3. Let n be a positive integer. For $0 \leq i \leq n$ denote by $\delta_i^n : [n-1] \to [n]$ the unique monotone injection which omits i from its range. This is the ith coface map. Concretely,

$$\delta_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

Also define $\sigma_i^n : [n+1] \to [n]$ to be the unique monotone surjection with $\sigma_i^n(i) = \sigma_i^n(i+1)$. This is the ith codegeneracy map. Concretely,

$$\sigma_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Geometrically, δ_i^n is the inclusion of the ith face of Δ^n (meaning the face opposite the ith vertex) and σ_n^i is the projection of Δ^{n+1} onto Δ^n where we collapse the edge $[e_i \; e_{i+1}]$ down to a point.

[1]But for those who are interested, there is a theory of unoriented "symmetric simplicial sets"

Any monotone map $f : [n] \to [m]$ has a decomposition into a surjection $[n] \twoheadrightarrow [k]$ and an injecton $[k] \hookrightarrow [m]$; this may be easiest to see if we think of $[k]$ as the image f with the order inherited from $[m]$ (passing to the category of all finite nonempty totally ordered sets). Furthermore the injection $[k] \hookrightarrow [m]$ can be decomposed into a composition of coface maps, omitting elements of $[m]$ one at a time, and the surjection $[n] \twoheadrightarrow [k]$ may be decomposed into a composition of codegeneracy maps, squishing together elements $i, i+1$ such that $f(i) = f(i+1)$ one at a time until none remain. This tells us that every morphism in Δ is a composition of coface and codegeneracy maps. In fact there is a normal form associated to this decomposition, obtained by repeatedly applying the "cosimplicial identites".

Theorem 2.4. *The simplex category Δ is the free category C on a sequence of objects $[0], [1], \dots$ and families of morphisms $\{\delta_i^n \in \mathrm{Hom}_{\mathsf{C}}(n-1, n)\}_{n\geq 1, 0\leq i\leq n}$ and $\{\sigma_i^n \in \mathrm{Hom}_{\mathsf{C}}(n+1, n)\}_{n\geq 0, 0\leq i\leq n}$, subject to the relations (for all n)*

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad (\textit{if } i < j) \tag{1}$$
$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_i^{n+1} \circ \sigma_{j-1}^n \quad (\textit{if } i < j) \tag{2}$$
$$\sigma_j^n \circ \delta_j^{n+1} = \mathrm{id}_{[n]} \tag{3}$$
$$\sigma_j^n \circ \delta_{j+1}^{n+1} = \mathrm{id}_{[n]} \tag{4}$$
$$\sigma_j^{n+1} \circ \delta_i^{n+2} = \delta_{i-1}^{n+1} \circ \sigma_j^n \quad (\textit{if } i > j+1) \tag{5}$$
$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \quad (\textit{if } i \leq j). \tag{6}$$

We will not prove this theorem in these notes, but we will attempt to explain what these identites say in the simplex category and explain what it means for a category to be presented by generators and relations. The equations (1) and (2) are a commutativity condition, they express (with index shifts appropriate to the δ's and σ's) that omitting a vertex i and then omitting/collapsing a later vertex j is the same as first omitting/collapsing $j - 1 = \delta_i^{-1}(j)$ and then omitting i. The equations (3) and (4) are perhaps the most important identities, because their categorical interpretation is that each δ is a *split monomorphism* and each σ is a *split epimorphism*; explicitly they say that if we omit a vertex and then collapse it with the next/previous vertex, it's the same as doing nothing. The equation (5) can be understood as saying "far away" omissions/collapses do not affect eachother (up to reindexing!). And finally equation (6) expresses that if you collapse twice in a row, the order of collapses matters only in that it shifts up the indexing.

The "free category" part of the theorem is more directly relevant, because it gives an explicit description of functors $\Delta \to \mathsf{C}$ for any category C (like how a presentation of a group G tells you what group homomorphisms $G \to H$ are). One interpretation of a "free structure" is exactly this kind of universal property, i.e. a free thing ("group" or "category equipped with a sequence of objects and families of maps satisfying the cosimplicial identities") is an initial object in the category of things. A free group G on generators $x_1, \dots, x_n$ subject to relations $r_1, \dots, r_m$ is an initial object in the category of tuples $(H, y_1, \dots, y_n)$ of groups H and $\mathbf{y} \in H^n$ such that for each j, interpreting x_i as y_i in w_j gives the identity element of H; a morphism $(H, \mathbf{y}) \to (H', \mathbf{z})$ in this category is of course a group homomorphism $f : H \to H'$ such that $f(y_i) = z_i$ for each i. Hence a free category on objects $\{X_s\}_{s\in S}$ and morphisms $\{f_\lambda : X_s \to X_t\}_{s,t\in S, \lambda\in\Lambda_{s,t}}$ subject to some equations of morphisms $\{E_j\}_{j\in J}$ is an initial object in the category[2] of categories that are equipped with a chosen family of objects labelled by S and a chosen family of morphisms labelled by the $\Lambda_{s,s'}$ satisfying all equations E_j. There is also a "by hand" construction of a free category on a directed graph/quiver G, e.g. the graph with vertices $\mathbb{N}$ and edges labelled by the coface/codegeneracy maps. This construction is fairly simply, if v, w are vertices in G then a morphism $v \to w$ in the free category is just a path ("formal composition of edges") from v to w in G. One can then quotient the set of arrows of this category by the smallest equivalence relation which contains the equations and "respects composition" (like how a normal subgroup gives an equivalence relation which multiplication).

We might stop and ask at this point why we need the codegeneracies at all. If we're interested in gluing together simplices along their boundaries, surely we just need the face inclusions? It turns out that the category of simplicial sets is much nicer when degeneracies; for example, the geometric realization of the semisimplicial set $\Delta^1 \times \Delta^1$ is an interval union two points, not a square! The theory without degeneracies isn't useless, though, we obtain what are called "semi-simplicial sets". These are called "Δ-complexes" in Hatcher's algebraic topology textbook.

We now return to simplicial sets, having gained an understanding of what kind of "formal simplices" we're gluing together. The categorical understanding of "gluing" is that it is a colimit. And vice versa, in many concrete categories a colimit does performing some kind of concrete "gluing". This is all there is to the definition of a simplicial set.

[2]Here I mean the locally small category of small categories such that etc. But in fact an initial object of this category will have the right mapping out property with respect to locally small categories too, as any functor $F : \mathsf{C} \to \mathsf{D}$ with C small factors through a small subcategory D' of D; specifically D' is the full subcategory of D on the objects in the image of F.

Definition 2.5. The category of simplicial sets $s\mathsf{Set}$ is the free[3] cocompletion of Δ. That is to say $s\mathsf{Set}$ has all (small) colimits, comes equipped with a functor $Y : \Delta \to s\mathsf{Set}$, and for any other category D with all (small) colimits and functors $F : \Delta \to \mathsf{D}$ there exists a colimit preserving functor $G : s\mathsf{Set} \to \mathsf{D}$ equipped with an isomorphism $\tau : F \to G \circ Y$. Furthermore (G, τ) is unique in that if we have another colimit-preserving functor $G' : s\mathsf{Set} \to \mathsf{C}$ equipped with an isomorphism $\tau' : F \to G' \circ Y$ then there is a unique isomorphism $\zeta : G \to G'$ making the diagram

$$\begin{tikzcd} & G \circ Y \arrow[d, "\zeta Y"] \\ F \arrow[ur, "\tau'"] \arrow[r, "\tau"'] & G' \circ Y \end{tikzcd}$$

commute.

Intuitively this says that an object of $s\mathsf{Set}$ is a formal colimit of some diagram in Δ. One can construct a free cocompletion in this way, but I tried to write it down once and lost two weeks working out technical details. Luckily the free cocompletion of a small category is a recognizable, fairly simple, and extremely well behaved category. The rest of this section will be devoted to proving the following theorem.

Theorem 2.6. *Let* C *be a small category and* $\mathrm{Psh}(\mathsf{C}) = \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ *be the category of presheaves on* C*. The Yoneda embedding* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *exhibits* $\mathrm{Psh}(\mathsf{C})$ *as the free cocompletion of* C*.*

Most people would find my initial definition of $s\mathsf{Set}$ a little silly. The true definition is just $s\mathsf{Set} = \mathrm{Psh}(\Delta)$. Our presentation of Δ tells us that a simplicial set can also be understood as sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with morphisms s_i^n, d_i^n satisfying the *simplicial identities*, the categorical dual of the cosimplicial identities (because presheaves are contravariant functors $\Delta \to \mathsf{Set}$). We will expand on this later.

One caveat with Theorem 2.6 is that, because the presheaf category is a free construction, already existing colimits in C will almost never be preserved under y. The proof of Theorem 2.6 boils down to the fact that any presheaf on C can be canonically written as a colimit of representable presheaves (those in the image of the Yoneda embedding). This may sound strange, but it's actually just another point of view on the celebrated Yoneda lemma, which we recall below.

Lemma 2.7. *Let* C *be a small category and* $y : \mathsf{C} \to \mathrm{Psh}(\mathsf{C})$ *the functor* $y(x) = \mathrm{Hom}_{\mathsf{C}}(-, x)$*. For any object* x *of* C *and presheaf* S *on* C *the function* $\varphi : \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \to S(x)$ *defined by* $\varphi(\eta) = \eta_x(\mathrm{id}_x)$ *is a bijection. Furthermore,* φ *defines a natural isomorphism of functors* $\mathsf{C}^{\mathrm{op}} \times \mathrm{Psh}(\mathsf{C}) \to \mathsf{Set}$*.*

For the rest of this section we use the notation φ as in this lemma and set $\psi = \varphi^{-1}$.

Proof. We define an inverse $\psi : S(x) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$ by $\psi(s)_z(f) = S(f)(s)$. Unwrapping this a bit, for $s \in S(x)$ we define a natural transformation $\psi(s) : y(x) \to S$ by setting its component on an object z to be the function $\mathrm{Hom}_{\mathsf{C}}(z, x) \to S(z)$ sending $f : z \to x$ to its action on s under S, i.e. $S(f)(s)$. We must verify that $\psi(s)$ is in fact natural for each s. So suppose we have a map $g : z \to w$ in C, we must check that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(w, x) \arrow[r, "g^*"] \arrow[d, "\psi(s)_w"] & \mathrm{Hom}_{\mathsf{C}}(z, x) \arrow[d, "\psi(s)_z"] \\ S(w) \arrow[r, "S(g)"] & S(z) \end{tikzcd}$$

commutes. By unravelling definitions and applying functoriality of S we calculate for any $f \in \mathrm{Hom}_{\mathsf{C}}(w, x)$ that

$$\psi(s)_z(g^*(f)) = \psi(s)_z(f \circ g) = S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(\psi(s)_w(f)).$$

So $\psi(s)$ is natural. And for arbitrary $s \in S(x)$, $\eta \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S)$, $z \in \mathrm{Obj}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(z, x)$ we have

$$\varphi(\psi(s)) = \psi(s)_x(\mathrm{id}_x) = S(\mathrm{id}_x)(s) = \mathrm{id}_{S(x)}(s) = s$$

$$\psi(\varphi(\eta))_z(f) = S(f)(\varphi(\eta)) = S(f)(\eta_x(\mathrm{id}_x)) \overset{(!)}{=} \eta_z(y(x)(f)(\mathrm{id}_x)) = \eta_z(f^*(\mathrm{id}_x)) = \eta_z(f).$$

[3] This is not quite freeness in the sense discussed above; it is about 2-initiality in an appropriate 2-category!

The equality labelled (!) holds because of the following naturality square of η:

$$\begin{tikzcd} y(x)(x) \arrow[r, "y(x)(f)"] \arrow[d, "\eta_x"] & y(x)(z) \arrow[d, "\eta_z"] \\ S(x) \arrow[r, "S(f)"] & S(z). \end{tikzcd}$$

This proves φ is a bijection. To check φ is natural it suffices to show it is natural in x for fixed S and natural in S for fixed x. Fix S and write φ_x for φ. We must show that for any morphism $a : u \to v$ in C the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(v), S) \arrow[r, "y(a)^*"] \arrow[d, "\varphi_v"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(u), S) \arrow[d, "\varphi_u"] \\ S(v) \arrow[r, "S(a)"] & S(u) \end{tikzcd}$$

commutes. This is once again just unfolding definitions and using naturality, as for any $\eta : y(v) \to S$ we calculate

$$\varphi_u(y(a)^*(\eta)) = \varphi_u(\eta \circ y(a)) = \eta_u(y(a)_u(\mathrm{id}_u)) = \eta_u(a \circ \mathrm{id}_u) = \eta_u(a)$$
$$S(a)(\varphi_v(\eta)) = S(a)(\eta_v(\mathrm{id}_v)) = \eta_u(y(v)(a)(\mathrm{id}_v)) = \eta_u(\mathrm{id}_v \circ a) = \eta_u(a).$$

Now fix x and write φ_S for S. Let $\beta : S \to T$ be an arbitrary natural transformation. The diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \arrow[r, "\beta_*"] \arrow[d, "\varphi_S"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), T) \arrow[d, "\varphi_T"] \\ S(x) \arrow[r, "\beta_x"] & T(x) \end{tikzcd}$$

commutes because for any $\eta : y(x) \to S$ we have

$$\varphi_T(\beta_*(\eta)) = \varphi_T(\beta \circ \eta) = \beta_x(\eta_x(\mathrm{id}_x)) = \beta_x(\varphi_S(\eta)).$$ □

So what does an isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S) \cong S(x)$ have to do with writing S as a colimit? The key point is that naturality in the x argument means that for any map $f : z \to w$ in C and $s \in S(w)$ we have

$$\varphi_{z,S}(\psi_{w,S}(s) \circ y(f)) = \varphi_{z,S}(y(f)^*(\psi_{w,S}(s))) = S(f)(\varphi_{w,S}(\psi_{w,S}(s))) = S(f)(s).$$

Hence for any $t \in S(z)$, $s \in S(w)$ and map $f : z \to w$ satisfying $S(f)(s) = t$ there is a commutative triangle

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\psi_{z,S}(t)"'] & y(w) \arrow[d, "\psi_{w,S}(s)"] \\ & S. \end{tikzcd}$$

These triangles suggest that S is a cocone under a certain diagram with structure maps $\psi_{w,S}(s) : y(w) \to S$. An object of the indexing category must know about both w and $s \in S(w)$ and a morphism has to be constrained by $S(f)(s) = t$.

Definition 2.8. Let C be a small category and S a presheaf on C. Define a (small) category $\mathrm{el}(S)$[4] by

$$\mathrm{Obj}(\mathrm{el}(S)) = \{(x, s) : x \in \mathrm{Obj}(\mathsf{C}), s \in S(x)\}$$
$$\mathrm{Hom}_{\mathrm{el}(S)}((z, t), (w, s)) = \{f \in \mathrm{Hom}_{\mathsf{C}}(z, w) : S(f)(s) = t\}.$$

We set $\mathrm{id}_{(x,s)} = \mathrm{id}_x$ and perform composition as in C. The identities are well defined because $S(\mathrm{id}_x)(s) = s$. The composition laws are automatic, and this composition is well defined because if $S(f)(s) = t$ and $S(g)(t) = r$ then

$$S(f \circ g)(s) = S(g)(S(f)(s)) = S(g)(t) = r.$$

This category comes with a forgetful functor $P_S : \mathrm{el}(S) \to \mathsf{C}$. The category $\mathrm{el}(S)$ equipped with P_S is referred to as the category of elements of S (in the special case $\mathsf{C} = \Delta$ it is sometimes called the category of simplices of S). It is instructive to think about what happens in the case that S is the forgetful functor of some familiar category, e.g. finite groups (or to make C small and not just essentially small, groups whose underlying set is hereditarily finite).

[4] A reader with stacky inclinations may recognize this as the grothendieck construction, specialized to presheaves of discrete groupoids (sets).

Theorem 2.9. *Let* C *be a small category and* S *a presheaf on* C*. The morphisms* $\psi_{x,S}(s) : y(x) \to S$ *make* S *into a colimit of the diagram* $y \circ P : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})$.

Proof. We already saw that these maps assemble into a cocone by naturality of the Yoneda lemma. Suppose we have a presheaf T on C and natural transformations $\sigma^{x,s} : y(x) \to T$ such that for any morphism $f : (z,t) \to (w,s)$ in $\mathrm{el}(S)$,

$$\begin{tikzcd} y(z) \arrow[r, "y(f)"] \arrow[dr, "\sigma^{z,t}"'] & y(w) \arrow[d, "\sigma^{w,s}"] \\ & T \end{tikzcd}$$

commutes. We are then required to show there is a unique natural transformation $\beta : S \to T$ making each diagram

$$\begin{tikzcd} y(x) \arrow[d, "\psi_{x,S}(s)"'] \arrow[dr, "\sigma^{x,s}"] & \\ S \arrow[r, "\beta"'] & T \end{tikzcd}$$

commute. Uniqueness is immediate, as naturality of the Yoneda lemma in the presheaf argument gives

$$\beta \circ \psi_{x,S}(s) = \beta_*(\psi_{x,S}(s)) = \psi_{x,T}(\beta_x(s))$$

and hence commutativity of the requisite triangles is equivalent to the identity $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$. So we just need to check that the maps $\beta_x(s) = \varphi_{x,T}(\sigma^{x,s})$ assemble into a natural transformation $S \to T$. This means that the square

$$\begin{tikzcd} S(w) \arrow[r, "S(f)"] \arrow[d, "\beta_w"] & S(z) \arrow[d, "\beta_z"] \\ T(w) \arrow[r, "T(f)"] & T(z) \end{tikzcd}$$

must commute for any $f : z \to w$ in C, which in turn is true because for $s \in S(w)$, abbreviating $t = S(f)(s)$, we have

$$T(f)(\beta_w(s)) = T(f)(\varphi_{w,T}(\sigma^{w,s})) = \varphi_{z,T}(y(f)^*(\sigma^{w,s})) = \varphi_{z,T}(\sigma^{w,s} \circ y(f)) = \varphi_{z,T}(\sigma^{z,t}) = \beta_z(t) = \beta_z(S(f)(s)). \quad \square$$

With Theorem 2.9 we have shown that $\mathrm{Psh}(\mathsf{C})$ is generated from $y(\mathsf{C}) \simeq \mathsf{C}$ under "gluing" (colimits). We now have the tools to prove Theorem 2.6, which states that this method of gluing objects of C together is universal. The reader may already see how to define a colimit-preserving extension of a functor $F : \mathsf{C} \to \mathsf{D}$ using the colimit formula for presheaves: send $S = \mathrm{colim}(y \circ P)$ to $\mathrm{colim}(F \circ P)$. But in set-theoretic foundations the term $\mathrm{colim}(F \circ P)$ isn't really meaningful; "$\mathrm{colim}(F \circ P)$" is only defined up to isomorphism, not equality. We do not have a canonical choice of colimit in D, and choosing an arbitrary one simultaneously across the proper class of presheaves S requires a stronger choice axiom than is in ZFC. In univalent mathematics there is no issue, since equality and isomorphism are the same thing. In ZFC+Grothendieck universes the "class" of presheaves is only a proper class from the point of view of some ambient inacessible cardinal κ. Our presheaves are valued in V_κ and so there is just a set of them, to which ZFC's axiom of choice applies. But this noncanonical choice is still awkward, so we opt to consider all choices without bias. However also in ZFC (or really NBG, so we can talk about classes) we can carry this argument out as long as the target category D has "explicitly defined" or "distinguished" colimits, hich happens in all situations we will see in these notes.

Fix a small category C, a locally small category D, and a functor $F : \mathsf{C} \to \mathsf{D}$.

Definition 2.10. A realization functor is a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ such that for every presheaf S on C, G preserves the colimit of the diagram $y \circ P_S$. Non-rigorously, G is a realization functor if for every S it satisfies the equation

$$G(S) = G\left(\underset{e \in \mathrm{el}(S)}{\mathrm{colim}}\, y(P_S(e))\right) = \underset{e \in \mathrm{el}(S)}{\mathrm{colim}}\, G(y(P_S(e))).$$

Say that G extends F if $G \circ y$ is isomorphic to F. In this case G must satisfy the (non-rigorous) identity

$$G(S) = \underset{e \in \mathrm{el}(S)}{\mathrm{colim}}\, F(P_S(e)).$$

Lemma 2.11. *In sufficiently strong foundations, if* D *has all small colimits then for every functor* $F : \mathsf{C} \to \mathsf{D}$ *there exists a realization functor* $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ *which extends* F*.*

Proof. We begin by correcting a deficiency from earlier, which was defining the category of elements as a *function* $\mathrm{Obj}(\mathrm{Psh}(\mathsf{C})) \to \mathrm{Obj}(\mathsf{Cat})$ instead of a *functor* $\mathrm{Psh}(\mathsf{C}) \to \mathsf{Cat}$. For a morphism of presheaves $\alpha : S \to T$ define $\mathrm{el}(\alpha) : \mathrm{el}(S) \to \mathrm{el}(T)$ by $\mathrm{el}(\alpha)(x, s) = (x, \alpha_x(s))$ on objects and $\mathrm{el}(\alpha)(f) = f$ on morphisms. This is well defined as

$$T(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t).$$

for any morphism $f : (z, t) \to (w, s)$. The functor laws for $\mathrm{el}(-)$ hold since $\mathrm{el}(\mathrm{id}_S)(x, s) = (x, (\mathrm{id}_S)_x(s)) = (x, s)$ and

$$\mathrm{el}(\beta \circ \alpha)(x, s) = (x, (\beta \circ \alpha)_x(s)) = (x, \beta_x(\alpha_x(s))) = \mathrm{el}(\beta)(x, \alpha_x(s)) = \mathrm{el}(\beta)(\mathrm{el}(\alpha)(x, s)).$$

Also note that for any $\alpha : S \to T$ we have $P_T \circ \mathrm{el}(\alpha) = P_S$ (the functor $\mathrm{el}(\alpha)$ leaves the first coordinate unchanged). By assumption we may choose for each S a colimit A_S of $F \circ P_S$, with structure maps $\kappa^{S,e} : F(P_S(e)) \to A_S$. Define a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ on objects by $G(S) = A_S$. For a natural transformation $\alpha : S \to T$ the equality $P_T \circ \mathrm{el}(\alpha) = P_S$ allows us to “pull back” the $(y \circ P_T)$-cocone structure on T along $\mathrm{el}(\alpha)$ to a $(y \circ P_S)$-cocone structure, the structure maps of which are $\kappa^{T,\mathrm{el}(\alpha)(e)} : F(P_S(e)) \to A_T$ (for e an object of $\mathrm{el}(S)$). Then since A_S is an initial cocone of $F \circ P_S$ there exists a unique morphism $G(\alpha) : A_S \to A_T$ such that for all $e \in \mathrm{Obj}(\mathrm{el}(S))$ the diagram

$$\begin{tikzcd}
F(P_S(e)) \arrow[d, "\kappa^{S,e}"'] \arrow[dr, "\kappa^{T,\mathrm{el}(\alpha)(e)}"] & \\
A_S \arrow[r, dashed, "G(\alpha)"'] & A_T
\end{tikzcd}$$

commutes. It is easy to verify the functor laws for G using this definition and functoriality of $\mathrm{el}(-)$; we leave this to the reader. So we have defined a functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. We prove $F \cong G \circ y$ and then that G is a realization functor.

To show G extends F we examine the structure of the category $\mathsf{E} = \mathrm{el}(y(x))$ for a general object x of C. The key observation is that (x, id_x) is a terminal object of E. For any other object (z, f) of E we have at least into our proposed terminal object, $f \in \mathrm{Hom}_{\mathsf{E}}((z, f), (x, \mathrm{id}_x))$. And an arbitrary map $g : (z, f) \to (x, \mathrm{id}_x)$ must satisfy $g^*(\mathrm{id}_x) = f$, hence $g = f$. It's a standard result that a diagram with indexing category which admits a terminal object has colimit the image of that terminal object. In parrticular the structure map $\kappa^{y(x),(x,\mathrm{id}_x)} : F(x) \to A_{y(x)}$ must be an isomorphism. Hence we can define a natural isomorphism $\tau : F \to G \circ y$ by $\tau_x = \kappa^{y(x),(x,\mathrm{id}_x)}$, as long as the square

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[d, "\tau_z"] & F(w) \arrow[d, "\tau_w"] \\
G(y(z)) \arrow[r, "G(y(f))"] & G(y(w))
\end{tikzcd}$$

commutes for each morphism $f : z \to w$ in C. Equivalently, $G(y(f)) = \tau_w \circ F(f) \circ \tau_z^{-1}$. By the definition of the action of G on morphisms and the equality $\mathrm{el}(y(f))(z, \mathrm{id}_z) = (z, f)$, this is equivalent to commutativity of

$$\begin{tikzcd}
F(z) \arrow[rr, "\kappa^{y(w),(z,f)}"] \arrow[d, "\kappa^{y(z),(z,\mathrm{id}_z)}"'] & & A_{y(w)} \\
A_{y(z)} \arrow[r, "\tau_z^{-1}"] & F(z) \arrow[r, "F(f)"] & F(w), \arrow[u, "\tau_w"]
\end{tikzcd}$$

which immediately reduces to the equation $\kappa^{y(z),(z,\mathrm{id}_z)} = \tau_w \circ F(f)$. But this equation is part of the cocone structure on $A_{y(w)}$, specifically the commuting triangle associated to the morphism $f : (z, f) \to (w, \mathrm{id}_w)$ in $\mathrm{el}(y(w))$.

Finally with τ in hand it is easy to show G is a realization functor. Let S be an arbitrary presheaf. We must show that the morphisms $G(\psi_{x,S}(s)) : G(y(x)) \to G(S)$ make $G(S) = A_S$ a colimit of $G \circ y \circ P_S$. It suffices to show that this is true after transporting the cocone structure across the isomorphism $\tau P_S : F \circ P_S \to G \circ y \circ P_S$, i.e. that the morphisms $G(\psi_{x,S}(s)) \circ \tau_x : F(x) \to A_S$ make A_S a colimit of $F \circ P_S$. By definition of the action of G on morphisms and the equality $\mathrm{el}(\psi_{x,S}(s))(x, \mathrm{id}_x) = (x, \psi_{x,S}(s)_x(x, \mathrm{id}_x)) = (x, \varphi_{x,S}(\psi_{x,S}(s))) = (x, s)$ we have

$$G(\psi_{x,S}(s)) \circ \tau_x = G(\psi_{x,S}(s)) \circ \kappa^{y(x),(x,\mathrm{id}_x)} = \kappa^{S,\mathrm{el}(\psi_{x,S}(s))(x,\mathrm{id}_x)} = \kappa^{S,(x,s)}$$

and so this $(F \circ P_S)$-cocone structure on A_S is originally chosen one, which we know is colimiting. $\square$

So in order to show Psh(C) is the free cocompletion of C it suffices to show that realization functors extending F are unique up to a unique isomorphism and that they preserve all colimits. We accomplish both by recasting the notion of a realization functor in terms of adjointness.

Definition 2.12. For any $F : \mathsf{C} \to \mathsf{D}$ the nerve of F is the functor $N(F) : \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$ defined by

$$N(F)(d)(x) = \mathrm{Hom}_{\mathsf{D}}(F(x), d).$$

Lemma 2.13. *For any realization functor $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ and isomorphism $\tau : F \to G \circ y$ there is an adjunction $G \dashv N(F)$ whose unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ satisfies, for each presheaf S and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$,*

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x.$$

Proof. For each presheaf S on C define $\eta_S : S \to N(F)(G(S))$ by $(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$. These assemble into a natural transformation, i.e. for any morphism $f : z \to w$ in C we have a commuting square

$$\begin{tikzcd}
S(w) \arrow[r, "S(f)"] \arrow[d, "(\eta_S)_w"] & S(z) \arrow[d, "(\eta_S)_z"] \\
\mathrm{Hom}_{\mathsf{D}}(F(w), G(S)) \arrow[r, "F(f)^*"] & \mathrm{Hom}_{\mathsf{D}}(F(z), G(S)).
\end{tikzcd}$$

To prove this square commutes, let $s \in S(w)$ be arbitrary and define $t = S(f)(s)$. By naturality of τ,

$$\begin{aligned}
F(f)^*((\eta_S)_w(s)) &= F(f)^*(G(\psi_{w,S}(s)) \circ \tau_w) \\
&= G(\psi_{w,S}(s)) \circ \tau_w \circ F(f) \\
&= G(\psi_{w,S}(s)) \circ G(y(f)) \circ \tau_z \\
&= G(\psi_{w,S}(s) \circ y(f)) \circ \tau_z \\
&= G(\psi_{z,S}(t)) \circ \tau_z \\
&= (\eta_S)_z(t) \\
&= (\eta_S)_z(S(f)(s)).
\end{aligned}$$

The reader might expect us to now verify η_S is natural in S and write down a counit, but if we use the "universal arrow" characterization of adjunctions this is unnecessary. What we do need to do to obtain an adjunction with (necessarily natural) unit $S \mapsto \eta_S$ is argue that for any object d of D and $\alpha : S \to N(F)(d)$ there exists a unique morphism $\beta : G(S) \to d$ in D such that $\alpha = N(F)(\beta) \circ \eta_S$. By calculating

$$(N(F)(\beta) \circ \eta_S)_x(s) = N(F)(\beta)_x((\eta_S)_x(s)) = \beta \circ (\eta_S)_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$$

we find that a morphism $\beta : G(S) \to d$ satisfies $\alpha = N(F)(\beta) \circ \eta_S$ iff $\alpha_x(s) = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$ for every object (x, s) of $\mathrm{el}(S)$. We can push forward the $(y \circ P_S)$-cocone structure on S along α to get a $(y \circ P_S)$-cocone structure on $N(F)(d)$, with structure maps $\alpha \circ \psi_{x,S}(s) : y(x) \to N(F)(d)$. Let $a^{x,s} = \varphi_{x,N(F)(d)}(\alpha \circ \psi_{x,S}(s))$, i.e. $a^{x,s} = \alpha_x(s)$. Then $a^{x,s} \in N(F)(d)(x)$, meaning $a^{x,s}$ is a morphism $F(x) \to d$ in D. In fact these morphisms make d into a cocone under $F \circ P_S$, i.e. for any map $f : (z, t) \to (w, s)$ in $\mathrm{el}(S)$ the diagram

$$\begin{tikzcd}
F(z) \arrow[r, "F(f)"] \arrow[dr, "a^{z,t}"'] & F(w) \arrow[d, "a^{w,s}"] \\
& d
\end{tikzcd}$$

commutes. This is by naturality of α and the equation $S(f)(s) = t$ (baked into the definition of a morphism $\mathrm{el}(S)$), as

$$a^{w,s} \circ F(f) = F(f)^*(a^{w,s}) = N(F)(d)(f)(a^{w,s}) = N(F)(d)(f)(\alpha_w(s)) = \alpha_z(S(f)(s)) = \alpha_z(t) = a^{z,t}.$$

We may transport this along τ to get the structure of a cocone under $G \circ y \circ P_S$ on d, with structure maps $a^{x,s} \circ \tau_x^{-1}$. Because G is a realization functor, $G(S)$ is a colimit of $G \circ y \circ P_s$ with structure maps $G(\psi_{x,S}(s))$. Thus there is a unqiue $\beta : G(S) \to d$ such that $a^{x,s} \circ \tau_x^{-1} = \beta \circ G(\psi_{x,S}(s))$, equivalently $a^{x,s} = \beta \circ G(\psi_{x,S}(s)) \circ \tau_x$, as desired. $\square$

Corollary 2.14. *If $G : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ is a realization functor then there is an adjunction $G \dashv N(G \circ y)$ with unit map $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(G \circ y) \circ G$ satisfying $(\eta_S)_x(s) = G(\psi_{x,S}(s))$. In particular realization functors preserve all colimits.*

Proof. Apply Lemma 2.13 with $\tau = \mathrm{id}_{G \circ y}$. □

A funny implication of Corollary 2.14 is that there is no difference between "realization functors" $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, colimit preserving functors $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$, and left adjoints $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$. With this we finally prove Theorem 2.6.

Proof. Let D be a cocomplete category and $F : \mathsf{C} \to \mathsf{D}$ any functor. Existence of a colimit-preserving functor $\mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ extending F is immediate from Lemmas 2.11 and Lemma 2.13. Suppose $G, G' : \mathrm{Psh}(\mathsf{C}) \to \mathsf{D}$ preserve all colimits and we have $\tau : F \cong G \circ y, \tau' : F \cong G' \circ y$. Then by Lemma 2.13 we have adjunctions $G \dashv N(F)$ and $G' \dashv N(F)$ with unit maps $\eta : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G$ and $\eta' : \mathrm{Id}_{\mathrm{Psh}(\mathsf{C})} \to N(F) \circ G'$ such that

$$(\eta_S)_x(s) = G(\psi_{x,S}(s)) \circ \tau_x$$
$$(\eta'_S)_x(s) = G'(\psi_{x,S}(s)) \circ \tau'_x$$

for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. By uniqueness of left adjoints there is a unique isomorphism $\zeta : G \to G'$ such that $\eta' = N(F)\zeta \circ \eta$. For any presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$ we may calculate

$$((N(F)\zeta \circ \eta)_S)_x(s) = ((N(F)\zeta)_S)_x((\eta_S)_x(s)) = N(F)(\zeta_S)_x((\eta_S)_x(s)) = \zeta_S \circ (\eta_S)_x(s) = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x.$$

So an isomorphism $\zeta : G \to G'$ satisfies $\eta' = N(F)\zeta \circ \eta$ iff $G'(\psi_{x,S}(s)) \circ \tau'_x = \zeta_S \circ G(\psi_{x,S}(s)) \circ \tau_x$ for each presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Additionally $\zeta_S \circ G(\psi_{x,S}(s)) = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)}$, so ζ is unique such that

$$G'(\psi_{x,S}(s)) \circ \tau'_x = G'(\psi_{x,S}(s)) \circ \zeta_{y(x)} \circ \tau_x$$

for every presheaf S on C and $(x, s) \in \mathrm{Obj}(\mathrm{el}(S))$. Taking $S = y(x)$ and $s = \mathrm{id}_x$ this reduces to $\tau'_x = \zeta_{y(x)} \circ \tau_x$, and conversely $\tau'_x = \zeta_{y(x)} \circ \tau_x$ implies the general case. Hence there is a unique iso $\zeta : G \to G'$ with $\tau' = \zeta y \circ \tau$. □

We finish the section by proving that any slice of a presheaf category is still a presheaf category. This enables us to argue by "base change"; the reader who has learned algebraic geometry understands how useful this is. Hopefully this elucidates why we're spending so much time on general presheaf categories if our ultimate interest is simplicial sets.

Lemma 2.15. *Let C be a small category and x an object of C. Then there is an isomorphism of categories between* $\mathrm{el}(y(x))$ *and the slice category C/x which commutes with their respective projections down to C.*
Furthermore, for any presheaf S on C there is an equivalence of categories $\mathrm{Psh}(\mathrm{el}(S)) \simeq \mathrm{Psh}(\mathsf{C})/S$ such that the composition $\mathrm{el}(S) \hookrightarrow \mathrm{Psh}(\mathrm{el}(S)) \to \mathrm{Psh}(\mathsf{C})/S$ is isomorphic to the cocone structure of S over $P_S \circ y$.[5]

Proof. The isomorphism $\mathrm{el}(y(x)) \cong C/x$ is so simple it might in fact be an equality (depending on how you define the slice category). An object (z, t) of $\mathrm{el}(y(x))$ just corresponds to the object $t : z \to x$ of C/x. Morphisms are the identified under this correspondence because if we have $s : z \to x$ and $t : w \to x$, a map $f : z \to w$ defines a morphism $(z, t) \to (w, s)$ iff $s \circ f = t$ iff it defines a map $s \to t$ in the slice category.

Now let S be a presheaf on C. Let $F : \mathrm{el}(S) \to \mathrm{Psh}(\mathsf{C})/S$ be the cocone structure of S over $y \circ P_S$, so its value on an object (x, s) is the object $\psi(s) : y(x) \to S$ in the slice category and it leaves morphisms unchanged. By the results of this section we have an adjunction $G \dashv N(F)$ such that $F \cong G \circ y$. We prove that G is an equivalence by showing it is fully faithful and essentially surjective (it follows from this that $N(F)$ is a quasi-inverse and the unit/counit of the adjunction $G \dashv N(F)$ are isomorphisms). In fact, we're first going to establish G is essentially surjective assuming it is fully faithful. Let E be the essential image of G and $i : E \hookrightarrow \mathrm{Psh}(\mathsf{C})/S$ the inclusion. Then G factors through i to give an equivalence $\mathrm{Psh}(\mathrm{el}(S)) \to E$, and so in particular E has all colimits and its inclusion i preserves them. Because colimits in slice categories are computed as in the original category, every object of $\mathrm{Psh}(\mathsf{C})/S$ is a colimit of representable presheaves (equipped with structure maps down to S), and hence to show G is essentially surjective it suffices to show E contains all objects of the form $y(x) \xrightarrow{v} S$ for x an object of C. But an object of this form lies in the image of F, which we know is contained in E.

[5]For a diagram $L : \mathsf{J} \to \mathsf{D}$ the structure of a cocone over L on an object d of D is the same thing as a lift of L to D/d.

Now we must show G is fully faithful, i.e. that its action $\mu_{A,B} : \mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A, B) \to \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A), G(B))$ is bijective for any presheaves A, B on $\mathrm{el}(S)$. For any morphisms $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : A' \to B$ there is an evident equality $G(\beta \circ \gamma \circ \alpha) = G(\beta) \circ G(\gamma) \circ G(\alpha)$, meaning we have a naturality square for μ

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A', B) \arrow[r, "\mathrm{Hom}(\alpha,\beta)"] \arrow[d, "\mu_{A',B}"] & \mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(A, B') \arrow[d, "\mu_{A,B'}"] \\
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A'), G(B)) \arrow[r, "\mathrm{Hom}(G(\alpha),G(\beta))"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(A), G(B')).
\end{tikzcd}$$

It suffices to show each "partially applied" $\mu_{-,B}$ is an isomorphism $\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(-, B) \cong \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(-), G(B))$. Because the contravariant Hom-functor preserves limits (with domain the opposite category, i.e. it sends colimits to limits) and G preserves colimits, both the domain and codomain of $\mu_{-,B}$ are limit preserving functors. The full subcategory of $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$ on objects A such that $\mu_{A,B}$ is an isomorphism is closed under limits in $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$, since naturality of $\mu_{-,B}$ and the fact that the functors it goes between both preserve limits imply that $\mu_{\lim_i A_i,B} = \lim_i \mu_{A_i,B}$, and a limit of isomorphisms is an isomorphism. But $\mathrm{Psh}(\mathrm{el}(S))$ is generated under colimits by representable functors, so $\mathrm{Psh}(\mathrm{el}(S))^{\mathrm{op}}$ is generated under limits by the same, and hence it suffices to show $\mu_{y(e),B}$ is an isomorphism for any object e of $\mathrm{el}(S)$. Now we play the same game and change our goal to proving that each natural transformation $\mu_{y(e),-}$ is an isomorphism, and similarly to before if we can argue that its domain and codomain both preserve colimits we reduce to showing that just the $\mu_{y(e_1),y(e_2)}$ is an isomorphism. This time we can't appeal to a general property of the covariant Hom functor, since it will not always preserve colimits. However the yoneda lemma tells us that $\mathrm{Hom}_{\mathrm{Psh}(\mathrm{el}(S))}(y(e), -)$ is isomorphic to the functor of evaluation at e, which preserves colimits. But we also know $G(y(e)) \cong F(e)$, and G preserves colimits, so for $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(G(y(e)), G(-))$ to preserve colimits it suffices for $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), -)$ to preserve colimits. Let $e = (x, s)$ and define $U : \mathrm{Psh}(\mathsf{C})/S \to \mathrm{Psh}(\mathsf{C})$ to be the forgetful functor. Similarly to how G being a functor makes μ a natural transformation, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right) \arrow[r, "\alpha_*"] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), S) \arrow[d, hook] \\
\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), T) \arrow[r, "\alpha_*"] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), S).
\end{tikzcd}$$

commutes. In fact it is a cartesian square, since $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), S) = \{\psi(s)\}$ and $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right)$ is *defined* to be the set of all $\sigma : y(x) \to T$ such that $\alpha_*(\sigma) = \psi(s)$. The conclusion that $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), -)$ preserves colimits then follows from three facts: the left hand morphism in our cartesian square is the component of a natural transformation $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), -) \hookrightarrow \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), U(-))$ at $T \xrightarrow{\alpha} S$, the functor $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})}(y(x), U(-))$ preserves colimits (since both U and evaluation at x do), and colimits in Set are pullback stable. This last term means that for any morphism $f : X \to Y$ of sets, the pullback functor $- \times_Y X : \mathsf{Set} \to \mathsf{Set}$ preserves colimits. So due to the various universal properties at play, the functor $\mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}\left(F(e), T \xrightarrow{\alpha} S\right)$ must preserve colimits.

Lastly we must verify $\mu_{y(e),y(e')}$ is an isomorphism for all objects e, e' of $\mathrm{el}(S)$. Because $G \circ y \cong F$ we reduce to checking that F is fully faithful. Let $e = (z, t)$ and $e' = (w, s)$ be objects of $\mathrm{el}(S)$. We have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{el}(S)}(e, e') \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), F(e')) \arrow[d, hook] \\
\mathrm{Hom}_{\mathsf{C}}(z, w) \arrow[r] & \mathrm{Hom}_{\mathsf{C}}(y(z), y(w))
\end{tikzcd}$$

where the vertical maps are subset inclusions, the top map is the action of F, and the bottom map is the action of y. We could also view the vertical maps as the actions of P_S and of U, from which perspective commutativity of this diagram holds because $U \circ F = y \circ P_S$. The bottom map is an isomorphism because the yoneda embedding is fully faithful, and so the top map is just the restriction of an isomorphism to certain subsets. Thus in order to show F is fully faithful it suffices to show that a map $f : z \to w$ satisfies $f \in \mathrm{Hom}_{\mathrm{el}(S)}(e, e')$ if and only if $y(f) \in \mathrm{Hom}_{\mathrm{Psh}(\mathsf{C})/S}(F(e), F(e'))$. Unravelling definitions this means $S(f)(s) = t$ iff $\psi_{w,S}(s) \circ y(f) = \psi_{z,S}(t)$. But this is something we've already used frequently, it's obvious from the definition of $\varphi_{-,S}$ (and the fact φ is an isomorphism). $\square$

3 Examples of simplicial sets

Lemma 2.13 is a powerful tool in understanding simplicial sets. It makes it incredibly easy to define an adjunction between simplicial sets and some other category C, you simply need a functor $\Delta \to \mathsf{C}$. This construction is crucial to understanding simplicial sets, and many important families of simplicial sets arise from adjunctions like this. Perhaps the most important adjunction of this kind, one which the reader has certainly seen before in disguise and which is the source of the term "realization", is the one induced by the functor $\Delta \to \mathsf{Top}$ in Definition 2.2.

Definition 3.1. The singular simplicial set of a topological space X is $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$. This has the structure of a functor as it is the nerve of the geometric simplex functor $\Delta \to \mathsf{Top}$. The geometric realization of a simplicial set S is defined by $|S| = \operatorname*{colim}_{\sigma \in \ell(S)} \Delta^n$. The colimit here is taken over the category $\mathrm{el}(S)$ from definition 2.8, i.e. $|S|$ is a realization functor extending the geometric simplex functor. But note that Top has an explicit choice of colimits, some sort of quotient space of a disjoint union space, so this $|\bullet|$ thing really is a single well defined functor.

The functor Sing occurs implicitly in the definition of singular homology, as the nth chain group of X is just the free abelian group on $\mathrm{Sing}(X)_n$. And furthermore, the differential of the singular chain complex is an alternating sum of the face maps of $\mathrm{Sing}(X)$! Note that the set $\mathrm{Sing}(X)$ is *massive*, for essentially any space X. The only simplicial sets we really know of at this point are the standard simplices $\Delta[n]$, i.e. the representable functors $y([n])$, which are finite in each degree and generated under the face and degneracy maps by finitely many simplices in total. On the other hand if X is a manifold or a CW complex of positive dimension then the set of maps $\Delta^0 \to X$ will be uncountable (it is in bijection with the underlying set of X). This is sort of a reflection of our original frustration with topological spaces: from the perspective of simplicial sets or other combinatorial models of homotopy types they have an enormous amount of redundant data.

The geometric realization functor seems incredibly opaque, but it can be surprisingly simple to calculate in practice. Most simplicial sets, even "small" ones, have infinitely many simplices, so the definition seems somewhat intractible for the purpose of actual computation. But because geometric realization is a left adjoint we know it preserves all colimits, and we know that the geometric realization of $\Delta[n]$ can be canonically identified with Δ^n. So if we have a "presentation" for a simplicial set as a finite colimit (or just simple to undestand colimit) of standard simplices then we get a description of its geometric realization as the same colimit but taken in Top. To illustrate this point we discuss two important families of simplicial sets and calculate their geometric realization. These families are motivated by simple geometric examples, and the presentation we give of them makes it clear their geometric realizations are in fact the topological spaces which model those geometric objects.

Definition 3.2. A simplicial subset of a simplicial set T is a sequence of sets $\{S_n\}_{n=0}^{\infty}$ such that $S_n \subseteq T_n$ for each n and $T(f)(S_n) \subseteq S_m$ for any morphism $f : [m] \to [n]$ in the simplex category (it suffices that this condition hold when f is a coface or codegeneracy map). The inclusions $S_n \subseteq T_n$ assemble into a monomorphism $S \hookrightarrow T$ in $s\mathsf{Set}$.

Definition 3.3. For any n define a simplicial subset $\partial\Delta[n]$ of $\Delta[n]$ by the formula

$$(\partial\Delta[n])_m = \{f \in \Delta[n]_m \mid f \text{ is not surjective}\}.$$

To check that this is well defined we must show that for any $g : [\ell] \to [k]$ we have $g^*((\partial\Delta[n])_k) \subseteq (\partial\Delta[n])_\ell$. But this is clear, because if $f \circ g$ is surjective then f must be as well.

Additionally, for any $0 \leq i \leq n$ define the ith "horn" of $\Delta[n]$ by

$$(\Lambda^n_i)_m = \{\alpha \in \Delta[n]_m \mid [n] \nsubseteq (\alpha([m]) \cup \{i\})\}.$$

The condition $[n] \nsubseteq (\alpha([m]) \cup \{i\})$ says that α must omit some vertex other than the ith, i.e. that α factors through some coface map $\delta_j : [n-1] \to [n]$ for $j \neq i$. So geometrically this is saying Λ^n_i is the union of all faces of $\Delta[n]$ except the ith. This is a simplicial subset for similar reasons to $\partial\Delta[n]$.

Lemma 3.4. *Let C be a small category in which every map factors as a split epimorphism followed by a monomorphism. Suppose S is a presheaf on C which admits a monomorphism into a representable presheaf on C. Define $\mathcal{M}$ to be the full subcategory of* $\mathrm{el}(S)$ *on pairs (w, s) such that $\psi_{w,S}(s)$ is a monomorphism. Then $\mathcal{M}$ is cofinal in* $\mathrm{el}(S)$*. Informally, S is the colimit of its representable subfunctors.*

Proof. Let $i : S \hookrightarrow y(x)$ be a monomorphism. Cofinality says that every object of $\mathrm{el}(S)$ admits a map into some object of $\mathcal{M}$, and that all choices of maps are related in a suitable way. Let $e = (z, t)$ be any object of $\mathrm{el}(S)$ and define $\mathcal{M}_e$ to be the full subcategory of $\mathcal{M}$ on objects $e' = (w, s)$ such that $\psi_{z,S}(t)$ factors through $\psi_{w,S}(s)$. Note that if $e' \in \mathcal{M}_e$ then there is a *unique* morphism $f : z \to w$ such that $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(f)$, since $\psi_{w,S}(s)$ is a monomorphism (and y is fully faithful). And $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(f)$ is equivalent to $S(f)(s) = t$, so if $e' \in \mathcal{M}_e$ there exists a unique morphism $e \to e'$. We argue that $\mathcal{M}_e$ has an initial object.

Since y is fully faithful we can write $i \circ \psi_{z,S}(t) = y(f)$ for a unique map $f : z \to x$ in C. By assumption, f admits a factorization $f = j \circ p$ where $p : z \to w$ is split epic and $j : w \to x$ is monic. Let $a : w \to z$ be a section of p and define $s = S(a)(t)$. Then $\psi_{w,S}(s) = \psi_{z,S}(t) \circ y(a)$. This implies

$$i \circ \psi_{w,S}(s) = i \circ \psi_{z,S}(t) \circ y(a) = y(f) \circ y(a) = y(f \circ a) = y(j \circ p \circ a) = y(j)$$

and in particular $i \circ \psi_{w,S}(s)$ is a monomorphism. Hence $\psi_{w,S}(s)$ is a monomorphism, i.e. $(w, s) \in \mathcal{M}$. To show $(w, s) \in \mathcal{M}_e$ we prove $\psi_{z,S}(t) = \psi_{w,S}(s) \circ y(p)$. We can check this after postcomposing with i since i is monic, and

$$i \circ \psi_{w,S}(s) \circ y(p) = y(j) \circ y(p) = y(f) = i \circ \psi_{z,S}(t)$$

so $(w, s) \in \mathcal{M}_e$. We claim that (w, s) is initial. Since every morphism in $\mathcal{M}$ is monic it suffices to show (w, s) admits some map to any other object (v, r) of $\mathcal{M}_e$, uniqueness is immediate. Let $f : z \to w$ be the unique morphism satisfying $S(f)(r) = t$. Then

$$S(f \circ a)(r) = S(a)(S(f)(r)) = S(a)(t) = s$$

which means $f \circ a$ defines a map $(w, s) \to (v, r)$ in $\mathrm{el}(S)$, as desired.

We leave verification of cofinality using this property of $\mathcal{M}_e$ to the reader. □

Lemma 3.4 gives us an easy description of all simplicial subsets of the standard simplices, and of their geometric realizations. If $i : S \hookrightarrow \Delta[n]$ is a monomorphism then representable subfunctors of S are the same thing (under postcomposition with i) as representable subfunctors of $\Delta[n]$ which factor through i. But by the yoneda lemma, representable subfunctors of $\Delta[n]$ are really the same thing as injections into $[n]$ in the simplex category, or even more concretely the data of S is a collection of faces (of any lower dimension) of the n-simplex. Furthermore since S is a functor and compositions of injective maps are injective, any face of a simplex in this collection must still lie in the collection. But this data, a collection of faces closed under taking further faces, is that of an abstract simplicial complex with (ordered) vertices $[n]$! On the other hand, if we have an abstract simplicial complex $L \subseteq 2^{[n]}$ then we can define a simplicial subset L' of $\Delta[n]$ by

$$L'_m = \{\sigma \in \Delta[n]_m : \operatorname{im} \sigma \in L\}.$$

Evidently our simplicial subsets $\partial\Delta[n]$ and Λ^n_i are of this form, where L is the simplicial complex defining the boundary of an n-simplex or the ith horn of an n-simplex. In fact, suppose we have a simplicial complex $L \subseteq 2^{[n]}$ and view it as a poset (category). For any element $\sigma \in L$ we can uniquely write $\sigma = \{i_0, i_1, \dots, i_m\}$ where $0 \le i_0 < i_1 < \dots < i_m \le n$ and the i_j define a morphism $F(\sigma) : [m] \to [n]$ in the simplex category (geometrically, σ is a face of the n-simplex and embedding the m-simplex as that face). Evidently $F(\sigma) \in L'_m$, since $\operatorname{im} F(\sigma) = \sigma$. And since $F(\sigma)$ is an injection we have $([m], F(\sigma)) \in \mathcal{M}$, where $\mathcal{M} \subseteq \mathrm{el}(L')$ is the subcategory from Lemma 3.4. Furthermore if $\sigma \subseteq \tau$ then $F(\tau)$ factors through $F(\sigma)$ (the formula is awkward to write down, but intuitively we're just saying you can include σ into τ and then τ into $\Delta[n]$). Thus F defines a functor $L \to \mathcal{M}$, and in fact an isomorphism (we can write down an inverse $\mathcal{M} \to L$ on objects by sending $([k], \sigma)$ to $\operatorname{im} \sigma$, and if f factors through g we have $\operatorname{im} f \subseteq \operatorname{im} g$, so this assignment is monotone/functorial). Finally within the poset L we have a cofinal poset consisting of the maximal faces and their pairwise intersections (since L is not directed we need the pairwise intersections for cofinality). The point of this is that from a simplicial subset S of $\Delta[n]$ we can explicitly define a simplicial complex L and write down S as a finite colimits of representable functors using L; furthermore, since geometric realization commutes with colimits, $|S|$ is just the geometric simplex complex L! Or even more precisely, the geometric realization of $i : S \hookrightarrow \Delta[n]$ is isomorphic in the arrow category of topological spaces to the closed subset inclusion of L as a geometric subcomplex of Δ^n.

Corollary 3.5. *If $n > 1$ then we have a coequalizer diagram*

$$\coprod_{0 \le i < j \le n} \Delta[n-2] \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{0 \le k \le n} \Delta[n-1] \xrightarrow{p} \partial\Delta[n]$$

in which p is induced by the face maps $y(\delta^n_k) : \Delta[n-1] \to \Delta[n]$ and on the (i, j)th copy of $\Delta[n-2]$ the map f includes into the ith copy of $\Delta[n-1]$ via δ^{n-1}_{j-1} and g includes into the jth copy of $\delta[n-1]$ via δ^{n-1}_i.

Corollary 3.6. *If $0 \leq \ell \leq n$ and $n > 1$ then we have a coequalizer diagram*

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2] \rightrightarrows \coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta[n-1] \longrightarrow \Lambda^n_\ell$$

in which all maps are the same as in Lemma 3.5, just with restricted codomain.

Hopefully this discussion convinces the reader that simplicial sets have some connection to more concrete geometric objects (finite simplicial complexes) and that, although defined very abstractly, the geometric realization of a simplex can often be computed in a reasonable way.

Interlude: Stuff I meant to have mentioned already

The definition of a simplicial set as a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$ has an obvious generalization to categories other than Set.

Definition 3.7. Let C be a category. The category of simplicial objects in C is $s\mathsf{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C})$, i.e. a simplicial object in C is a functor $\Delta^{op} \to \mathsf{C}$. For a simplicial object X we often write X_n to abbreviate $X([n])$.

Due to Theorem 2.4 we can equivalently think of simplicial objects as diagrams of the form

$$\cdots \; X_2 \;\; X_1 \;\; X_0$$

which satisfy the formal duals of the cosimplicial identities, i.e. the simplicial identities

$$d^n_i \circ d^{n+1}_j = d^n_{j-1} \circ d^{n+1}_i \quad (\text{if } i < j) \tag{7}$$
$$d^{n+2}_i \circ s^{n+1}_j = s^n_{j-1} \circ d^{n+1}_i \quad (\text{if } i < j) \tag{8}$$
$$d^{n+1}_j \circ s^n_j = \mathrm{id}_{[n]} \tag{9}$$
$$d^{n+1}_{j+1} \circ s^n_j = \mathrm{id}_{[n]} \tag{10}$$
$$d^{n+2}_i \circ s^{n+1}_j = s^n_j \circ d^{n+1}_{i-1} \quad (\text{if } i > j+1) \tag{11}$$
$$s^{n+1}_i \circ s^n_j = s^{n+1}_{j+1} \circ s^n_i \quad (\text{if } i \leq j). \tag{12}$$

Here the s^*_* are the degree increasing maps in the diagram and the d^*_* are the degree decreasing ones, where the upper index is the degree of the source object and the lower index is the "height" in the stack of all maps with the same source and target. We call the s^*_* degeneracy maps of X and the d^*_* the face maps of X. In this point of view on simplicial objects, a morphism $X \to Y$ is a sequence of morphisms $X_k \to Y_k$ in C which intertwine the face and degeneracy maps of X with the face and degeneracy maps of Y. This is formally similar to the definition of connective chain complex, which is also a sequence of objects with structure maps and where a morphism between such is a degreewise morphism intertwining the structure maps. In fact, if C is abelian (or even just Ab-enriched) we can extract a connective chain complex of objects of C from a simplicial object X of C.

Definition 3.8. Let A be a category whose Hom-sets are abelian groups and where composition is $\mathbb{Z}$-bilinear. Given a simplicial object X of A the alternating face map complex of X is a nonnegatively graded chain complex C with $C_n = X_n$ and differential $\partial_n : C_n \to C_{n-1}$ defined by

$$\partial_n = \sum_{j=0}^{n} (-1)^j d^n_i.$$

This defines a functor $s\mathsf{A} \to \mathrm{Ch}_+(\mathsf{A})$ which is the identity on underlying $\mathbb{N}$-graded objects.

Exercise: Use the simplicial identities to prove $\partial_{n-1} \circ \partial_n = 0$ for any $n > 1$.

For example, if $S : \Delta^{\text{op}} \to \mathsf{Set}$ is a simplicial set then we can postcompose S with the free abelian group functor $\mathsf{Set} \to \mathsf{Ab}$ to obtain a simplicial abelian group. When $S = \operatorname{Sing}(X)$ for a topological space X, the alternating face map complex of this resulting simplicial abelian group is the singular chain complex of X. This demonstrates how natural constructions on simplicial sets can take us to the category of simplicial objects in another category. But sset is special; we've seen that it has a universal property in terms of extending functors defined on Δ, but also $s\mathsf{Set}$ is just a very nice category in its own right. Limits and colimits in functor categories are computed in the target category, so sSet has all small limits and colimits and these can be explicitly described by the formulas

$$\left(\operatorname*{colim}_i S^i\right)_n = \operatorname*{colim}_i\,(S^i)_n$$
$$\left(\lim_i S^i\right)_n = \lim_i\,(S^i)_n.$$

Furthermore the ith face/degeneracy map of the (co)limit is the (co)limit of the ith face/degeneracy map. Since the property of being an epi/monomorphism can be restated as a certain square being cocartesian/cartesian, an epimorphism of simplicial sets is a map which is surjective in each degree and a monomorphism of simplicial sets is a map which is injective in each degree. We can even do things like talk about an equivalence relation on a simplicial set S, meaning a simplicial subset of $S \times S$ which is degreewise an equivalence relation, and take quotients by these which have the correct universal property. The category $s\mathsf{Set}$ is a *Grothendieck topos* (of presheaf type!). One consequence of this is that $s\mathsf{Set}$ is cartesian closed, meaning that for any simplicial set S the product $- \times S$ has a right adjoint. Or stated in a more exciting way, we can talk about the simplicial set of morphisms between two simplicial sets! For simplicial sets S, T we wish to define a new simplicial set $\underline{\operatorname{Hom}}(S,T)$ such that $\operatorname{Hom}_{s\mathsf{Set}}(R \times S, T) \cong \operatorname{Hom}_{s\mathsf{Set}}(R, \underline{\operatorname{Hom}}(S,T))$ for any R (in a suitably natural way). Then taking $R = \Delta[n]$ we're forced to conclude

$$\underline{\operatorname{Hom}}(S,T)_n \cong \operatorname{Hom}_{s\mathsf{Set}}(\Delta[n], \underline{\operatorname{Hom}}(S,T)) \cong \operatorname{Hom}_{s\mathsf{Set}}(\Delta[n] \times S, T)$$

and so we might as well define $\underline{\operatorname{Hom}}(S,T)_n = \operatorname{Hom}_{s\mathsf{Set}}(\Delta[n] \times S, T)$. Note this expression is $\operatorname{Hom}_{s\mathsf{Set}}(y(-) \times S, T)$ applied to the object $[n]$ of sset, so we can get the full functorial structure of the simplicial set by defining

$$\underline{\operatorname{Hom}}(S,T) = \operatorname{Hom}_{s\mathsf{Set}}(y(-) \times S, T).$$

Via the adjunction between product and internal Hom we can define things like an "evaluation map" $\underline{\operatorname{Hom}}(S,T) \times S \to T$ (in fact this is the counit of the adjunction). This is one way that simplicial sets are nicer than topological spaces; for general topological spaces the function space won't have nice properties, and even if we restrict to some nicer subcategory of Top (e.g. compactly generated weak hausdorff spaces) we'll have to modify the usual definition of the product of two spaces to get the categorical product. Also, even the function space between two one-dimensional spaces is going to be infinite dimensional, while exponentials of "small" simplicial sets will still be fairly "small".

Back to section 3

We give one more example of a family of simplicial sets and then move on to homotopy theory. This family once again comes from a nerve-realization adjunction: the category Δ is defined as a subcategory of Pos (the category of posets) and Pos embeds in Cat, the category of small categories. So we have a functor $\Delta \to \mathsf{Cat}$ sending $[n]$ to the category

$$[0] \to [1] \to \ldots \to [n].$$

We will also denote this category by $[n]$.

Definition 3.9. The nerve of a small category C is the simplicial set $N(\mathsf{C})_n = \operatorname{Fun}([n], \mathsf{C})$, and taking the nerve is a right adjoint functor.

Note that a functor $[n] \to \mathsf{C}$, i.e. an n-simplex of the nerve of C, is really just a chain of n end-to-end (or "composable") morphisms in C. For $n = 0$ this degenerates to say $N(\mathsf{C})_0$ is the set of objects of C, and for $n = 1$ it says $N(\mathsf{C})_1$ is the set of morphisms of C. If we work out what the degeneracy and face maps look like from this perspective, it turns out that $d_1^0, d_0^0 : N(\mathsf{C})_1 \to N(\mathsf{C})_0$ are the source and target functions from arrows to objects and $s_0^0 : N(\mathsf{C})_0 \to N(\mathsf{C})_1$ sends an object to its associated identity map. In higher degrees, d_0^n and d_n^n just drop the first or last morphism in the sequence while an "inner" face map $d_i^n : N(\mathsf{C})_n \to N(\mathsf{C})_{n-1}$ for $0 < i < n$ composes the ith and $(i+1)$st morphism in a chain. The degeneracy maps $s_i^n : N(\mathsf{C})_n \to N(\mathsf{C})_{n+1}$ insert identity maps. More precisely, given a chain $\mathbf{f} = (f_1, \dots, f_n)$

$$x_0 \overset{f_1}{\to} x_1 \to \cdots \to x_{n-1} \overset{f_n}{\to} x_n$$

the chain $d_i^n(\mathbf{f})$ is

$$x_0 \overset{f_1}{\to} x_1 \to \cdots \to x_{i-1} \overset{f_{i+1}\circ f_i}{\to} x_{i+1} \to \cdots \to x_{n-1} \overset{f_n}{\to} x_n$$

while the chain $s_i^n(\mathbf{f})$ is

$$x_0 \overset{f_1}{\to} x_1 \to \cdots \to x_{i-1} \overset{f_i}{\to} x_i \overset{\mathrm{id}_{x_i}}{\to} x_i \overset{f_{i+1}}{\to} x_{i+1} \to \cdots \to x_{n-1} \overset{f_n}{\to} x_n.$$

In the case that C is a one object groupoid with finitely many morphisms, i.e. a finite group G, the space $|N(G)|$ is a classifying space BG for principal G-bundles. I.e., for any sufficiently nice space X homotopy classes of maps $X \to |N(G)|$ are in natural bijection with the isomorphism classes of principal bundles on X. There is a close connection between the group cohomology of G and sheaf cohomology of the classifying space of G (for a G-module M with trivial G-action, the group cohomology of G with coefficients in M is isomorphic to the singular cohomology of BG with coefficients in M). The simplicial set $N(G)$ is also easier to describe than the case of a general category, even if G is just a monoid and not a group, because all morphisms are composable. So $N(G)_n$ can be identified with G^n and the face maps either delete the first/last element of a tuple or multiply adjacent inner elements, while the degeneracy maps insert the identity of G in a suitable position.

This analysis of the face/degeneracy maps in particular tells us that $N(\mathsf{C})$ includes all the data that the category C. The 0 and 1-simplices tell us the sets of objects and arrows, the degree 0 degeneracy tells us the identity morphism, and the face map d_1^2 is the composition operation on pairs of composable arrows.S Since a morphism $f : N(\mathsf{C}) \to N(\mathsf{D})$ must commute with the face and degeneracy maps of the nerve, in fact f_0, f_1 define a functor $\mathsf{C} \to \mathsf{D}$. So N is not just a functor from categories to simplicial sets, it is an embedding of categories (i.e., is fully faithful)! Additionally since N preserves products (it is a right adjoint), and the natural map $N(\mathrm{Fun}(\mathsf{C}, \mathsf{D})) \to \underline{\mathrm{Hom}}(N(\mathsf{C})N(\mathsf{D}))$ is an isomorphism. To give an example, we can identify $\Delta[1] \times \Delta[1]$ with the poset having Hasse diagram

$$\begin{tikzcd} (0,1) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0). \arrow[u] \end{tikzcd}$$

4 Homotopy of simplicial sets

It can sometimes be helpful to think of $\mathrm{Sing}(X)$ as the archetypal simplicial set. That is, we think of a simplicial set S as encoding a space and an element $\sigma \in S_n$ as an n-dimensional simplex within that space (but one which might be highly degenerate). This perspective is enabled by the Yoneda lemma: for any simplicial S we have a natural isomorphism $S_n \cong \mathrm{Hom}_{s\mathsf{Set}}(\Delta[n], S)$. Under this perspective, a face map $d_i^n : S_n \to S_{n-1}$ is literally sending a simplex to its ith face and a degeneracy map $s_i^n : S_n \to S_{n+1}$ just relabels an n-simplex as a degnerate simplex of higher dimension. Of course the simplices in an arbitrary simplicial set are not as well behaved as the simplices that make up a simplicial complex; by considering the behavior $\mathrm{Sing}(X)$ one sees that in general a simplex is not determined by its faces and that the faces of a simplex do not need to be distinct. In fact, due to the presence of degeneracy maps, for any nonempty simplicial set and any i there is a simplex whose ith and $(i+1)$st faces are equal.

However, thinking of an arbitrary simplicial set as behaving like $\mathrm{Sing}(X)$ can also be dangerous. Continuous maps are very flexible and morphisms of simplicial sets are not (that was our whole motivation for defining simplicial sets!). If we try to think of a simplicial set geometrically, e.g. if we define it by drawing a picture, it may have deceptively few simplices. Compare the "interval" $\Delta[1]$, equipped with its basepoints $\delta_1^1, \delta_0^1 : \Delta[0] \to \Delta[1]$, to the topological interval

$I = [0, 1] \cong |\Delta[1]|$, with its basepoints $x_0, x_1 : \{*\} \to I$. In Top we have a commutative diagram

$$\begin{tikzcd}
& & \{*\} \arrow[d, "x_1"'] \arrow[dd, bend left, "x_1"] \\
& \{*\} \arrow[r, "x_0"'] \arrow[d, "x_1"] & I \arrow[d, "f"'] \\
\{*\} \arrow[r, "x_0"] \arrow[rr, bend right, "x_0"'] & I \arrow[r, "g"] & I \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the map f compresses I into the right half subinterval and the map g compresses I into the left half subinterval, and the square inside this diagram is cocartesian. In essence this is saying that I is a fractal: you can bisect it into two copies of itself. We then have a natural bijection $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to \mathrm{Hom}_{\mathsf{Top}}(I, X) \times_X \mathrm{Hom}_{\mathsf{Top}}(I, X)$ for a topological space X, where the maps $\mathrm{Hom}_{\mathsf{Top}}(I, X) \to X$ we're taking a pullback with respect to spit out the initial and terminal endpoints of a path. Invertibility of this map is why we can concatenate paths in a topological space (and since homotopies are the same as paths in the function space, at least for nice enough spaces, this is also why homotopy of maps is an equivalence relation). This entire story breaks down horribly for the $\Delta[1]$. The set $\mathrm{Hom}_{s\mathsf{Set}}(\Delta[1], \Delta[1]) \cong \mathrm{Hom}_{\Delta}([1], [1])$ has exactly three elements: the identity, the constant map at the first vertex of $\Delta[1]$, and the constant map at the second vertex of $\Delta[1]$. Our interest in simplicial sets is that they are more combinatorial, more discrete, but this means $\Delta[1]$ has no hope of being a fractal. We do still have a functor $s\mathsf{Set} \to \mathsf{Set}$ sending a simplicial set X to the set of "end to end" paths in X, i.e. $X \mapsto X_1 \times_{X_0} X_1$, and this is corepresented by the pushout of $\Delta[1]$ with itself over $\Delta[1]$, as in Top. But this pushout is the horn Λ^2_1, not $\Delta[1]$.

Because of this we're going to need to restrict our attention to some subclass of simplicial sets which is better suited for doing homotopy theory. What desiderata do we have for this class of homotopically nicer simplicial sets? Well, hopefully we should be able to define the fundamental group of these, or thinking more categorically the fundamental group*oid*. In particular the relation of being connected by a path (i.e. being isomorphic within the fundamental groupoid) should be an equivalence relation on vertices. For any simplicial set S we can define a relation $v \sim w$ on vertices $v, w \in S_0$ to mean there exists an edge $e \in S_1$ such that $d^1_1(e) = v$ and $d^1_1(e) = w$. As we've been intimating, this relation will not be an equivalence relation for an arbitrary S.

> Exercise: Let $e_0, e_1, \ldots, e_n$ denote the vertices of $\Delta[n]$. Show that $e_i \sim e_j$ if and only if $j = i$ or $j = i + 1$.

The exercise above gives examples where $\sim$ fails to be both symmetric and transitive. But it will always be reflexive, due to the presence of degenerate simplices. That is, for any $v \in S_0$ we have an edge $e = s^0_0(v) \in S_1$ satisfying both $d^1_1(e) = v$ and $d^1_1(e) = v$, hence $v \sim v$. So our first desideratum is that $\sim$ should be transitive on S, i.e. elements of S_1 should somehow be concatenable. There's not going to be an actual concatenation operation $S_1 \times_{S_0} S_1 \to S_1$, due to the differences between $\Delta[1]$ and the topological interval we described earlier. To figure out the right notion of concatenation for paths in a simplicial set we're going to return to topological spaces and analyze concatenation of paths in these more carefully. For a topological space X and paths $p, q : I \to X$ in X with $q(0) = p(1)$ we can define their concatenation by the formula

$$(p \bullet q)(t) = \begin{cases} p(2t) & \text{if } t \leq 0.5 \\ q(2t - 1) & \text{if } t \geq 0.5. \end{cases}$$

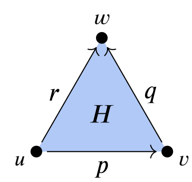
This is a concatenation of p and q in the sense that $\mathrm{im}(p \bullet q) = \mathrm{im}\, p \cup \mathrm{im}\, q$, but of course paths (even injective ones) have more information than just their images. In topology a path is a *parameterized* curve, and the restriction of $p \bullet q$ to $[0, 0.5]$ isn't going to be exactly the same as p because it has twice the velocity! Of course we don't really care about the velocity of a path, or particularly care about a parameterization at all, for the purpose of doing homotopy theory. Any nice reparameterization of a path (e.g. by precomposing with an orientation preserving homeomorphism of the interval) will not change the path-homotopy class. So our definition of $p \bullet q$ above is very very noncanonical, we could just as easily have defined it by e.g.

$$(p \bullet' q)(t) = \begin{cases} p\left(\frac{3}{2}t\right) & \text{if } t \leq \frac{2}{3} \\ q(3t - 2) & \text{if } t \geq \frac{2}{3}. \end{cases}$$

And of course, the operation $p \bullet q$ is not associative. If we had another path $r : I \to X$ with $r(0) = q(1)$ then we could concatenate the triple p, q, r in two different ways, either $(p \bullet q) \bullet r$ or $p \bullet (q \bullet r)$. Visually these are

$$(p \bullet q) \bullet r \qquad\qquad p \bullet (q \bullet r)$$

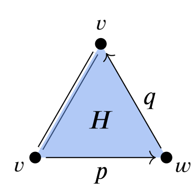
This isn't a failure of our specific definition of $\bullet$, the operation $\bullet'$ isn't associative either. There's just no way to define a binary concatenation operation on paths which both represents the correct class in the fundamental groupoid (i.e. is path-homotopic to $p \bullet q$) and is strictly associative. Of course, the two concatenations $(p \bullet q) \bullet r$ and $p \bullet (q \bullet r)$ are path-homotopic to eachother. This is the reason why multiplication in the fundamental group (or more properly composition in the fundamental groupoid) is associative! If we want to be homotopically unbiased, given two paths p, q in X with $p(1) = q(0)$ what we should say is that a concatenation of p and q is a path r in X equipped with a path-homotopy between $p \bullet q$ and r. We can even state this condition without privileging the concatenation $p \bullet q$ by asking for a triangle shaped homotopy in X, i.e. a map $H : \Delta^2 \to X$ whose restrictions to the edges of Δ^2 are p, q, and r. This definition has a clear generalization to simplicial sets: given edges $p, q \in S_1$ with $u = d_1^1(p), v = d_0^1(p) = d_1^1(q), w = d_0^1(q)$ a concatenation of p and q is an edge $r \in S_1$ and a 2-simplex $H \in S_2$ such that $d_2^2(H) = p$, $d_0^2(H) = q$, and $d_1^2(H) = r$.



We can encode the pair end-to-end edges p, q as a map $f : \Lambda_1^2 \to S$ and then 2-simplices H with $d_2^2(H) = p$ and $d_0^2(H) = q$ correspond under the yoneda lemma to maps $g : \Delta[2] \to X$ making the diagram

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r, "f"] \arrow[d, hook] & S \\
\Delta[2] \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

commute. So a sufficient condition for $\sim$ to be a transitive relation on S_0 is that the dashed arrow always exists in a diagram as above. If we want $\sim$ to be symmetric then any edge in S needs to be somehow invertible. With our understanding that degenerate 1-cells are like constant loops and concatenation of paths is defined by lifting diagrams as above there's an easy definition of inverses. For a path $p \in S_1$ with initial vertex $v = d_1^1(p)$ and terminal vertex $w = d_0^1(p)$ a right[6] inverse of p should be some $q \in S_1$ with $d_1^1(q) = w$ and $d_0^1(q) = v$ and such that $s_0^0(v)$ is a concatenation of p and q. That is, a right inverse of p is an edge $q \in S_1$ such that there exists a 2-simplex $H \in S_2$ with $d_2^2(H) = p$, $d_0^2(H) = q$, and $d_1^2(H) = s_0^0(v)$. When drawing simplices inside of a simplicial set it's common to use the symbol $=$ to denote a degnerate edge; with that convention we can depict this situation as



Phrased another way, p is right invertible iff there is a 2-simplex $\Delta[2] \to S$ which has $\overline{01}$ edge p and $\overline{02}$ edge $\sigma_0^0(v)$. For

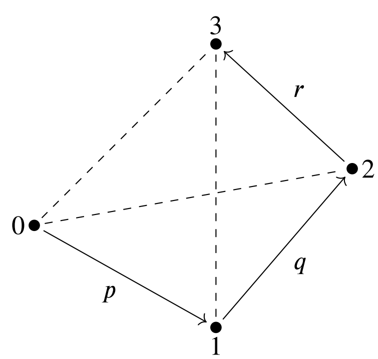
[6] Note that the order of arguments in path concatenation is opposite the order of composition of functions/composition in a category (in our conventions) and hence a "right inverse" of a path p is a path q such that travelling along p and then q gives the identity.

any p we can define a map $L_p : \Lambda^2_0 \to S$ with $\overline{01}$ edge p and $\overline{02}$ edge $\sigma^0_0(v)$, and p is right invertible iff in the diagram

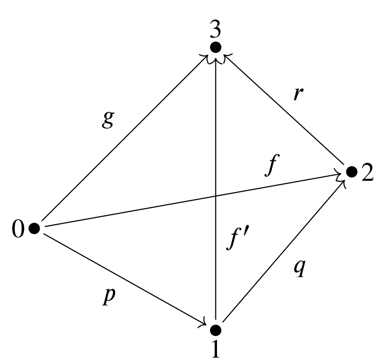
$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "L_p"] \arrow[d, hook] & S \\ \Delta[2] \arrow[ur, dashed] & \end{tikzcd}$$

there exists a dashed map making the triangle commute. There is an obvious dual notion of left invertibility, where we define $R_p : \Lambda^2_2 \to S$ to have $\overline{12}$ edge p and $\overline{02}$ edge $\sigma^0_0(v)$, and existence of left inverses says this map can be extended to a map $\Delta[2] \to S$ for any p. Either one of these assumptions is sufficient to ensure $\sim$ is symmetric, so we could arbitrarily say a simplicial set is S good for homotopy if any map $\Lambda^2_1 \to S$ has a filler and any map $\Lambda^2_0 \to S$ with degenerate $\overline{02}$ edge has a filler (by "has a filler" we mean "can be extended to a map out of the 2-simplex"). Of course, this is a terrible definition. It's ugly (although many definitions for simplicial sets are), nonsymmetric (and not just in appearance, these conditions are not strong enough to imply any horn $\Lambda^2_2 \to S$ with degenerate $\overline{02}$ edge has a filler), and most critically it doesn't say anything about the higher homotopy information of the space S encodes.

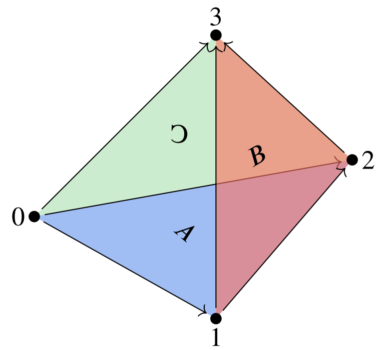
In fact, this definition only gives us a good definition of $\pi_0(S) = S/\sim$. It's not even strong enough to imply $\Pi_1(S)$ is a groupoid (or even that $\pi_1(S, v)$ is a group for any vertex $v \in S_0$). Before we can (attempt to) define the fundamental group(oid) we need a notion of homotopy of paths. We define this in sort of a silly way: for a path $p \in S_1$ from $v = d^1_1(p)$ to $w = d^1_0(p)$ we say p is homotopic to another path $q \in S_1$ iff q is a concatenation of p with $\sigma^0_0(w)$. Certainly this should be true after passing to homotopy classes if p and q are homotopic, and if we can make a groupoid out of homotopy clases of paths in which $\sigma^0_0(x)$ is the identity at $x \in S_0$ then this relation should also imply p and q have the same homotopy class, i.e. are homotopic. But there is once again asymmetry! We could instead have defined this relation by saying q is a concatenation of $\sigma^0_0(v)$ with p. And while reflexivity is clear (by applying an appropriate degeneracy to p) both symmetry and transitivity are totally opaque. Well even if we figured out how to solve all of these problems there would still be a fundamental issue. The group(oid) operation should be associative, which in our language means that if we have end-to-end-paths $p, q, r \in S_1$, f is a concatenation of p and q, and f' is a concatenation of q and r, any concatenation of f with r should be a concatenation of p with f' (and vice-versa). There is no way to ensure this with just fillers of 2-horns. Essentially the issue is that the three edges p, q, r want to live inside a three dimensional simplex, not a 2d one. They naturally form the "spine" of $\Delta[3]$, as depicted below.



Let g be a concatenation of f and r. We can then fill in the dashed arrows above and get the 1-skeleton of a tetrahedron.



If $A, B, C \in S_2$ are 2-simplices witnessing that f is a concatenation of p, q, that f' is a concatenation of q, r, and that g is a concatenation of f, r (respectively) then we can fill this diagram in even further.



This picture can be interpreted as a single morphism $\Lambda^3_2 \to S$, encoding all our data $p, q, r, f, f', g, A, B, C$. If we were able to fill this in to a 3-simplex $\Delta[3] \to S$ then the filler of the missing face $\overline{013}$ would exhibit g as a concatenation of f' with p, which is exactly the associativity condition we wanted. In the dual situation where we have a concatenation g' of p with g and construct a morphism $\Lambda^3_1 \to S$ the face $\overline{023}$ of a filling would exhibit g' as a concatenation of f with r. And in fact, filling 3-horns *is* sufficient for quite a lot of things to work out. The homotopy relation becomes transitive, the "left-biased" and "right-biased" path-homotopy relations become equivalent, and concatenation of paths is well defined and associative associative up to path-homotopy. Hence if S has fillers for all in horns in dimensions ≤ 3 we have a well defined "fundamental groupoid" $\Pi_1(S)$ with object set S_0, morphisms path-homotopy classes of paths, identity morphisms the degenerate 1-cells, and composition the (reversed) concatenation operation. This definition is not quite good enough to do proper homotopy theory, though! We're still missing necessary information to define the higher homotopy groups. Similar to the situation when we were stuck one dimension lower, while these assumptions are sufficient to ensure homotopy is an equivalence relation on paths it's not enough to ensure there is a way to concatenate homotopies themselves (in a manner well defined and associative up to 2-homotopy, etc). But there's a pretty clear generalization of what we've thought about to arbitrary dimensions.

Definition 4.1. A Kan complex is a simplicial set K such that for any $n > 0$ and $0 \leq i \leq n$ all morphisms $\Lambda^n_i \to X$ can be extended to a morphism $\Delta^n \to X$. That is, there always exists a dashed arrow making the diagram below commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Kan complexes are the simplicial sets with which it is suitable to do homotopy theory. One easy to define class of Kan complexes are the simplicial sets of the form $\mathrm{Sing}(X)$ for a topological space X. This can be seen by the adjunction $|-| \dashv \mathrm{Sing}$, which lets us restate the Kan lifting condition for $\mathrm{Sing}(X)$ in terms of lifting continuous maps $|\Lambda^n_i| \to X$ to maps $\Delta^n \to X$, and the fact that $|\Lambda^n_i|$ is a retract of Δ^n in Top.

There are two important variants on the definition of a Kan complex, called "weak Kan complexes" and "Kan fibrations". The first comes from the observation that our proof of transitivity of path connectedness only used fillers for the horn Λ^2_1, and the proof of associativity only used fillers for Λ^3_1 and Λ^3_2. In fact this is also true for the (omitted) proofs of transitivity of homotopy, well definedness of concatenation up to homotopy, and the coincidence of left and right biased homotopy. The only place we need to fill "outer horns" instead of "inner horns" is to establish the existence of left/right inverses of edges in a Kan complex. Thus the "fundamental groupoid" construction goes through if we're only able to fill the "inner horns" Λ^2_1, Λ^3_1, and Λ^3_2; but this won't generally be a groupoid, only a category.

Definition 4.2. A weak Kan complex is a simplicial set Q such that for any $0 < i < n$ all morphisms $\Lambda^n_i \to X$ can be extended to a morphism $\Delta^n \to X$. That is, there always exists a dashed arrow making the diagram below commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & Q \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

For the purpose of modelling the classical homotopy theory of topological spaces the "directedness" of simplicial sets seems like a deficiency, e.g. we have no symmetry operation $\Delta[1] \to \Delta[1]$ interchanging the vertices and so no uniform way of defining inverses across all Kan complexes. But it's precisely this directedness that allows us to encode categories as simplicial sets via the nerve construction, as we saw earlier. And while the nerve of a category is only a Kan complex if that category is a groupoid, the nerve is always a weak Kan complex! Weak Kan complexes are not intended to model homotopy types but *infinity categories*, which are inherently directed. In fact weak Kan complexes have many other names: quasicategories, ∞-categories, $(\infty, 1)$-categories, or even "homotopy theories".

The other direction of generalization is to consider families of Kan complexes. The term family is used here in the same sense as how a vector bundle is a family of vector spaces over the base, or how a morphism of schemes is the family of its fibers. I.e., by a family of Kan complexes we mean some sort of morphism of simplicial sets $f : E \to B$ whose fibers $\{E \times_B \{b\}\}_{b \in B_0}$ are Kan complexes. But not all morphisms will do, just like how not every continuous map with a vector space structure on all fibers is a vector bundle. We don't just want to lift horns in each fiber separately but have some sort of compatibility between the fillers, or even lift horns spread across many fibers.

Definition 4.3. A Kan fibration is a morphism of simplicial sets $f : E \to B$ such that for any $n > 0$ and $0 \le i \le n$, given any commutative square of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & E \arrow[d, "f"] \\ \Delta[n] \arrow[r] & B \end{tikzcd}$$

there exists a dashed map making the square below commute

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & E \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

If $E = \mathrm{Sing}(X)$, $B = \mathrm{Sing}(Y)$, and $f = \mathrm{Sing}(q)$ for a continuous map $q : X \to Y$ then f is a Kan fibration if and only if q is a Serre fibration, which is once again an application of the $|-| \dashv \mathrm{Sing}$ adjunction as well as carefully chosing a homeomorphism $[0,1] \times \Delta^{n-1} \cong \Delta^n$ that restricts to a homeomorphism between $\{0\} \times \Delta^{n-1}$ and $|\Lambda^n_i|$.

Theorem 4.4. *There is a model structure on the category $s\mathsf{Set}$, called the Quillen model structure, in which the cofibrations are the monomorphisms, the fibrations are the Kan fibrations, and the weak equivalences are the morphisms which become weak equivalences of topological spaces after taking geometric realization. This model structure is cofibrantly generated, with generating cofibrations the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ and generating acyclic cofibrations the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$.*

Proof. We will not give a proof here, simply highlight some approaches one could take and give exposition. There are a couple things to note about the theorem as stated:

- Our definition of weak equivalences is somewhat "cheap" in that it requires us to relate things back to topological spaces and a primitive notion of homotopy there, rather than giving a purely combinatorial definition of the model structure (which would be preferable, since simplicial sets are purely combinatorial objects). We could define weak equivalences in simplicial sets as in topolgoical spaces, using a combinatorial definition of homotopy groups—finding such a combinatorial definition of homotopy groups was Kan's original motivation for the theory of simplicial sets—but as we discussed above, even the naively defined fundamental group or pointed set of path components of a simplicial set at a vertex behaves poorly unless that simplicial set is a Kan complex. So what we might do is define a (functorial) fibrant replacement $F : s\mathsf{Set} \to s\mathsf{Set}$ and then declare a map φ to be a weak equivalence iff its $F(\varphi)$ is a weak equivalence between Kan complexes. In fact, the "cheap" definition arises in this way with replacement functor $F(X) = \mathrm{Sing}(|X|)$.

- In Quillen's original paper he took a fully combinatorial approach, in fact defining the model structure on Top as transferred from the one on $s\mathsf{Set}$. Key to his approach is the notion of a "minimal fibration".

- There is no particularly simple definition of acyclic cofibrations in this model structure other than the class generated by horn inclusions under standard model category operations (transfinite composition, pushouts, coproducts, retracts, etc). They are called "anodyne maps".

- It is not immediately obvious that monomorphisms in $s\mathsf{Set}$ are generated boundary inclusions, but also not that hard to prove. The intuitive picture is that for any simplicial subset $Y \subseteq X$, the set X can be obtained from Y by "attaching cells" (i.e., iteratively taking pushouts against $\partial\Delta^n \to \Delta^n$). We can, roughly, take this property (cofibrations = monomorphisms) as the fundamental input to the construction of the model structure and generate the rest of the data to be compatible with it. This is workable because $s\mathsf{Set}$ is a very nice category (a "topos") and furthermore Δ is a very nice category (a "test category"). This approach fits the Quillen model structure into Grothendieck's program of "test categories" (developed by Cisinski). After setting up the broader theory one obtains a model structure on $s\mathsf{Set}$ rather foramlly, and the only nontrivial work to be done is showing that the fibrations generated in this way coincide with the Kan fibrations. Cisinski's construction uses a combinatorially defined fibrant replacement functor over $\mathrm{Sing}(|\bullet|)$, known as "Kan's Ex^∞ functor". This functor is much better behaved and more explicit than $\mathrm{Sing}(|\bullet|)$, all it requires is the notion of (barycentric) subdivision and sequential colimits of simplicial sets. The Cisinski model structure approach is nice in that it generalizes to give a Quillen model structures on cubical sets, gives the Joyal model structure on simplicial sets (whose fibrant objects are weal Kan complexes rather than Kan complexes), gives with slight modification dendroidal sets (these model higher operads), and in general shows how to produce a model structure for presheaves on a category of "geometric shapes" like simplices, cubes, trees, etc.
- In Quillen's original approach the the notion of a minimal fibration brings in a dependence on the axiom of choice and the small object argument produces a fairly nonconstructive factorization. Cisinski's approach is more constructive, in particular it gives explicit factorizations using the Ex^∞ functor. But there is still a dependence on the law of the excluded middle squirreled away in the fact that, classically, any inclusion of sets $i : A \to B$ exhibits B as the coproduct of A and $B \setminus A$; if we take this instead as a property of the map i, a so-called *decidable inclusion*, then the combinatorial construction of the model structure goes through with little change. For some this is philosophically satisfying, but more practically it allows us to do homotopy theory with simplicial objects in a category of not-quite-sets. For example, this construction gives an model structure on the category of simplicial objects of $\mathrm{Sh}(X)$ for any topological space X. □

5 Derived Functors, Quillen Adjunctions, & Transferred Model Structures

Let M be a model category with cofibrations $\mathcal{C}$, fibrations $\mathcal{F}$, and weak equivalences $\mathcal{W}$. Recall that $\mathrm{Ho}(\mathsf{M}) = \mathsf{M}[\mathcal{W}^-1]$. The weak equivalences of M describe some sort of "homotopy theory" of the objects of M while cofibrations and fibrations are extra data used to control the localization at the weak equivalences (really the ∞-localization!). (Co)fibrations are also used to make sense of "homotopical constructions" in M, eg mapping cones, suspension, loop spaces. They're not the main thing and there's some redundancy in the data: either the class of cofibrations or of fibrations determines the other ($\mathcal{C} = \mathrm{LLP}(\mathcal{F} \cap \mathcal{W})$ and $\mathcal{F} = \mathrm{RLP}(\mathcal{C} \cap \mathcal{W})$). Perhaps the most important sort of "homotopical constructions" is that of a derived functor. The theory of derived functors really only requires a model structure on the domain category of a functor, we can work with a weaker notion for the codomain category.

Definition 5.1. A category with weak equivalences is a pair $(\mathsf{N}, \mathcal{V})$ where N is a category and $\mathcal{V}$ a class of maps in N which contains all isomorphisms and satisfies the 2-out-of-3 property.

Obviously a model category is such a thing, and we extend the notation $\mathrm{Ho}(\mathsf{N}) = \mathsf{N}[\mathcal{V}^{-1}]$ to the case where N is merely a category equipped with weak equivalences. Suppose M is a model categories, $(\mathsf{N}, \mathcal{V})$ a category equipped with weak equivalences, and $F : M \to N$ a functor. A derived functor of F is supposed to be some sort of "extension" of F to a functor $\mathrm{Ho}(M) \to \mathrm{Ho}(N)$. The existence of an extension in the usual sense, i.e. a functor $\overline{F} : \mathrm{Ho}(M) \to \mathrm{Ho}(N)$ making the square

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d] & N \arrow[d] \\ \mathrm{Ho}(M) \arrow[r, "\overline{F}"] & \mathrm{Ho}(N) \end{tikzcd}$$

commute up to natural isomorphism, is equivalent to the statement that F preserves weak equivalences (if F does so then the derived functor exists by the universal property of localization). Category theorists have a more flexible notion of extending a functor, called a (pointwise) Kan extension, and while this does give a good notion of a derived functor we choose to set it aside for the moment in favor of a more explicit analysis of when and how a functor can be derived.

Let M_f be the full subcategory of M on fibrant objects and $\mathcal{W}_f = \mathcal{W} \cap \mathsf{M}_f$ the class of weak equivalences between fibrant objects; define $\mathsf{M}_c, \mathcal{W}_c$ similarly but for cofibrant. The inclusion functor $i : \mathsf{M}_f \to \mathsf{M}$ satisfies $i(\mathcal{W}_f) \subseteq \mathcal{W}$, hence it induces a functor $\mathsf{M}_f[\mathcal{W}_f^- 1] \to \mathrm{Ho}(\mathsf{M})$. This latter functor is essentially surjective due to the existence of fibrant replacement. Less obvious is that it's fully faithful, but the factorization axioms of a model category give us enough control over localization to see that it is and hence is an equivalence of categories. Similarly $\mathsf{M}_c \to \mathsf{M}$ induces an equivalence $\mathsf{M}_c[\mathcal{W}_c^- 1] \to \mathrm{Ho}(\mathsf{M})$. The punchline is that if $F : M \to N$ preserves weak equivalences *between fibrant objects* then $F \circ i$ descends to the localizations, giving the right derived functor $\mathbb{R}F : \mathrm{Ho}(\mathsf{M}) \cong \mathsf{M}_f[\mathcal{W}_f^- 1] \to \mathrm{Ho}(\mathsf{N})$ of F. Similarly if $F(\mathcal{W}_c)$ are all weak equivalences we get a left derived functor $\mathbb{L}F : \mathrm{Ho}(\mathsf{M}) \cong \mathsf{M}_c[\mathcal{W}_c^- 1] \to \mathrm{Ho}(\mathsf{N})$.

Definition 5.2. Let M be a model category and N a category with a class of weak equivalences satisfying 2-out-of-3. A functor $F : \mathsf{M} \to \mathsf{N}$ is left-derivable if it sends weak equivalences between cofibrant objects to weak equivalences. Dually F is right-derivable if it sends weak equivalences between fibrant objects to weak equivalences.

This is how (total) derived functors in the sense of homological algebra fit into the framework of model categories. Any additive functor preserves chain homotopy, and weak equivalences between projective/injective complexes are automatically chain homotopy equivalences. The "resolution" point of view comes from the fact that cofibrant replacement (i.e. projective resolution) defines a functor $\mathrm{Ho}(\mathsf{M}) \to \mathsf{M}_c[\mathcal{W}_c^- 1]$ quasi-inverse to the inclusion. In fact right or left exactness isn't needed to derive the functor or to get a long exact sequence, we just need it to make sense of that LES as an extension of a one sided exact sequence. What general conditions ensure that a functor preserves weak equivalences between fibrant/cofibrant objects (in the nonabelian context)? The key tool is "Ken Brown's Lemma".

Lemma 5.3. *Let M be a model category and N a category equipped with weak equivalences. If a functor $F : \mathsf{M} \to \mathsf{N}$ sends trivial fibrations between fibrant objects to weak equivalences then it is right-derivable, F sends arbitrary weak equivalences between fibrant objects to weak equivalences.*

Proof. Let $w : X \to Y$ be a weak equivalence between fibrant objects. Factor the map $(\mathrm{id}_X, w) : X \to X \times Y$ into a trivial cofibration $i : X \to Z$ followed by a fibration $p : Z \to X \times Y$. The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are both fibrations since they are the base changes of the fibrations $Y \to 1$, $X \to 1$ along eachother. Let $f = \pi_1 \circ p$ and $g = \pi_2 \circ p$; note both f and g are fibrations (since they are both the composition of two fibrations). Also $f \circ i = \pi_1 \circ (p \circ i) = \pi_1 \circ (\mathrm{id}_X, w) = \mathrm{id}_X$ and $g \circ i = \pi_2 \circ (p \circ i) = \pi_2 \circ (\mathrm{id}_X, w) = w$ so in particular $f \circ i$ and $g \circ i$ are both weak equivalences. Since i is a weak equivalence the 2-out-of-3 axiom in M implies f and g are weak equivalences, hence trivial fibrations, hence $F(f)$ and $F(g)$ are weak equivalences. The equation $f \circ i = \mathrm{id}_X$ also implies $F(f) \circ F(i) = \mathrm{id}_{F(X)}$, hence by 2-out-of-3 in N the map $F(i)$ is a weak equivalence. And finally the equation $g \circ i = w$ implies $F(w)$ is a composition of two weak equivalences and hence is a weak equivalence, as desired. □

We have a good definition of "left-derivable" and "right-derivable" functors, and Lemma 5.3 gives a simpler criterion for checking these conditions. But these functors don't behave well when considered all together: since a "right-derivable" functor doesn't have to preserve fibrant objects, a composite of two right-derivable functors may fail to be right-derivable.

The notions of derivability we've found do their job and ensure a functor is derivable, but they're poorly suited categorically. Part of this is the asymmetry in the definition, as they require the domain to have more structure than they require of the codomain. Two left-derivable functors between model categories may not have left-derivable composition, as a left-derivable functor has no reason to send cofibrant objects to cofibrant objects. Taking a broader view, while derivability is all well and good it doesn't ask that a functor preserve as much structure as we might want for a "homomorphism of model categories".

Definition 5.4. Let M, N be model categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is left-Quillen if it preserves cofibrancy of objects, the class of cofibrations, and the class of trivial cofibrations. Dually F is right-Quillen if it preserves fibrancy of objects, the class of fibrations, and the class of trivial fibrations.

By Ken Brown's Lemma, a left-Quillen functor is automatically left-derivable and a right-Quillen functor is automatically right-derivable. Intuitively a left-Quillen functor preserves constructions in a model category that rely on the "cofibration" part of the data while a right-Quillen functor preserves constructions in a model category that rely on the "fibration" part of the data. Frequently it is assumed that a left-Quillen functor is cocontinuous and a right-Quillen functor is continuous; under this assumption (or the weaker one that initial/terminal objects are preserved) preservation of (co)fibrancy of objects follows from preservation of the class of (co)fibrations. Left and right Quillen functors frequently come in adjoint pairs, called "Quillen adjunctions". In some sense this is because of the following:

Lemma 5.5. *Let* M, N *be model categories and* $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ *an adjunction. The following are equivalent:*

(i) *the left adjoint* F *is left-Quillen,*

(ii) *the right adjoint* G *is right-Quillen,*

(iii) *the left adjoint* F *preserves cofibrations while the right adjoint* G *preserves fibrations,*

(iv) *the left adjoint* F *preserves trivial cofibrations while the right adjoint* G *preserves trivial fibrations.*

Proof. The key idea is that there is a correspondence between lifting problems of the left form in M and lifting problems of the right form in N

$$\begin{tikzcd} X \arrow[r, "s^\sharp"] \arrow[d, "p"'] & G(X') \arrow[d, "G(q)"] \\ Y \arrow[r, "t^\sharp"] & G(Y') \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "s^\flat"] \arrow[d, "F(p)"'] & X' \arrow[d, "q"] \\ F(Y) \arrow[r, "t^\flat"] & Y'. \end{tikzcd}$$

Here the flat-sharp notation says e.g. $s^\sharp$ and $s^\flat$ correspond under the bijection $\mathrm{Hom}_{\mathsf{M}}(X, G(X')) \cong \mathrm{Hom}_{\mathsf{N}}(F(X), X')$. Not only is there a correspondence between *problems*, but also *solutions*, i.e. between commuting diagrams

$$\begin{tikzcd} X \arrow[r, "s^\sharp"] \arrow[d, "p"'] & G(X') \arrow[d, "G(q)"] \\ Y \arrow[r, "t^\sharp"] \arrow[ur, dashed, "h^\sharp"] & G(Y') \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "s^\flat"] \arrow[d, "F(p)"'] & X' \arrow[d, "q"] \\ F(Y) \arrow[r, "t^\flat"] \arrow[ur, dashed, "h^\flat"] & Y'. \end{tikzcd}$$

It is straightforward to show from this that if M is endowed with classes of maps $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{L} = \mathrm{LLP}(\mathcal{R})$ and $\mathrm{RLP}(\mathcal{L}) = \mathcal{R}$ and N an analogous pair $(\mathcal{L}', \mathcal{R}')$ then for an adjunction $F \dashv G$ we have $F(\mathcal{L}) \subseteq \mathcal{L}'$ iff $\mathcal{R} \supseteq G(\mathcal{R}')$. $\square$

Lemma 5.6. *Let* $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ *be an Quillen adjunction. Then there is an adjunction* $\mathbb{L}F \dashv \mathbb{R}G$.

Proof. In fact, the adjunction doesn't need to be Quillen for the conclusion of the theorem, any adjunction of a left derivable and right derivable functor will do. Maltsiniotis gave an elegant proof using only formal properties of absolute/pointwise kan extensions. But the proof we present here requires these hypotheses.

For $X \in \mathrm{Obj}(\mathsf{M})$ cofibrant and $Y \in \mathrm{Obj}(\mathsf{N})$ fibrant there's a bijection $\mathrm{Hom}_{\mathsf{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathsf{N}}(F(X), Y)$. Additionally, $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(X, G(Y))$ can be identified with the quotient of $\mathrm{Hom}_{\mathsf{M}}(X, G(Y))$ by the homotopy relation on maps (which is an equivalence relation, since we're mapping from a cofibrant object to a fibrant one) and similarly $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{N})}(F(X), Y)$ can be identified with the quotient of $\mathrm{Hom}_{\mathsf{N}}(F(X), Y)$ by the homotopy relation on maps. Furthermore because X is cofibrant we can identify $(\mathbb{L}F)(X) = X$ because Y is fibrant we can identify $(\mathbb{R}G)(Y) = Y$. We then just need to descend the bijection $\mathrm{Hom}_{\mathsf{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathsf{N}}(F(X), Y)$ to a bijection of quotient sets $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{M})}(X, (\mathbb{R}G)(Y)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathsf{N})}((\mathbb{L}F)(X), Y)$, i.e. show that the original bijection respects homotopy of maps. And for this we can use either the fact that F sends good cylinder objects of X to good cylinder objects of $F(X)$ or the dual result for G, Y. $\square$

This proof doesn't just give an abstract adjunction, it tells us that the adjunct of a representative of a map is the representative of the adjunct of the map, when mapping from an M-cofibrant object to a N-fibrant object. For general derivable adjunctions there is still a nice relationship between the derived adjunction and the original adjunction, but it must be stated in terms of the unit and counit.

We now turn our eye to the problem of transferring a model structure on one category across an adjunction to another. Model structures are famously hard to construct, so it would be nice if we could do the hard work of verifying the existence of a single model structure (e.g. simplicial sets or chain complexes) and then bootstrap lots of others from that. The model case we have in mind is transferring the model structure on simplicial sets to a model structure on simplicial objects in some "algebraic" category A. The term "algebraic" here should be understood in the sense of universal algebra, so we include the cases of groups, abelian groups, rings, lie algebras, modules over a ring, etc, but not fields. The issue with fields is that we can't produce a "free field" on some set, whereas it's possible to do so for all other examples. So A should at least come with an adjunction $F : \mathsf{Set} \rightleftarrows \mathsf{A} : U$. Furthermore, free algebras have a sort of "finiteness" property to them, in that they are generated under *finitely many applications of* the algebraic operations by the basis elements. This is why $k[[X]]$ is not free on X, as infinitary operations are required to generate the ring. One way we could capture this finiteness property is that for any $t \in F(X)$ there is some finite subset $X' \subseteq X$ with inclusion $i : X' \to X$ such that $t \in \operatorname{im} F(i)$. So $U(F(X))$ should be the union of $\operatorname{im} U(F(i))$ over all inclusions i of finite sets. We can view this as U preserving a certain sort of colimit: the set X is a directed union of all its finite subsets, and this colimit will be preserves by the left adjoint F, and we're then asking for it to be preserved again by U. It turns out in fact that "monadic" adjunctions $F : \mathsf{Set} \rightleftarrows \mathsf{A} : U$ where the forgetful functor U preserves directed colimits are *exactly* the same as (multi-sorted) varieties of algebras in the sense of universal algebra. And such an adjunction lifts to an adjunction $F' : s\mathsf{Set} \rightleftarrows s\mathsf{A} : U'$ where the right adjoint U' still preserves directed colimits, because colimits in functor categories are pointwise and the functors F', U' are defined pointwise. So we take this as the basis for our theory of transfer.

Definition 5.7. Let N be a model category and M an ordinary category, with an adjunction $F : \mathsf{N} \rightleftarrows \mathsf{M} : U$. The right transferred model structure on M, if it exists, is the model structure in which a map φ is a weak equivalence iff $U(\varphi)$ is a weak equivalence in N and a map p is a fibration iff $U(p)$ is a fibration in N.

The reader may wonder why we pull back fibrations instead of cofibrations. The answer is that this adjunction "wants to be" a Quillen adjunction, in which case the fibrations should push forward to fibrations along the right adjoint. Or more generally, in passing lifting problems across an adjunction we should be taking the image of the right vertical map along the right adjoint. But there's also a good algebraic motivation for this. In categories of algebras, surjections are very well behaved in that they present the codomain as a quotient of the domain. Injections are less well behaved, e.g. subrings of Noetherian rings need not be Noetherian. In a model structure we often think of the fibrations as "nice surjections" and cofibrations as "nice injections", although this is not really true in general, so it makes sense for fibrations to take primacy over cofibrations. We could also consider the dual problem of pulling back a model structure along a left adjoint, and while the definition goes through the theory is not as nice (in some sense because we prioritize cofibrantly generated model structures over fibrantly generated ones).

Note that the data of fibrations and weak equivalences totally determines the model structure (if it exists) since the cofibrations are then determined as the maps with the left lifting property with respect to the trivial fibrations. However there are two possible definitions of a trivial cofibration in the situation of a transferred model structure, it could either be a map with the left lifting property with respect to all fibrations or as a map which is both a cofibration and a weak equivalence. We refer to the first sort of map as "anodyne" and the second as a "cofibration weak equivalence". If the transferred model structure exists then these two must coincide, but that is not the case in general. Anodyne maps are in particular cofibrations, since trivial fibrations are in particular fibrations, but they have no reason to be weak equivalences in general.

Example 5.8. Let k be a field of positive characteristic. Then there is an adjunction $\mathrm{Sym} : \mathrm{Ch}_+(k) \rightleftarrows \mathsf{CDGA}_+(k) : U$ and we've seen how to define a model structure on the codomain. If transferred model structure on $\mathsf{CDGA}_+(k)$ existed then this would be a Quillen adjunction, hence the left adjoint would preserve weak equivalences between cofibrant objects. That is to say, if $\varphi : P \to Q$ were a quasi-isomorphism between complexes of projectives then $U(\mathrm{Sym}(\varphi))$ would have to still be a weak equivalence. But Sym does not even send the generating acyclic cofibration $i : 0 \to D(n)$ to a weak equivalence. Since $\mathrm{Sym}(0) \cong k$ (in degree 0), to see that $\mathrm{Sym}(i)$ is not a weak equivalence it suffices to exhibit

It is a fundamental problem that anodyne maps need not be weak equivalences, and one that requires insight of the problem at hand to solve. But it turns out that this is the "only" obstruction, at least when the model category being transferred from is nice and the adjunction is "finitary" in the sense described earlier.

Lemma 5.9. *Let N be a cofibrantly generated model category and M an ordinary category with all small colimits. Given an adjunction $F : \mathsf{N} \rightleftarrows \mathsf{M} : U$, if U preserves directed colimits then any morphism in M factors as (1) a cofibration followed by a trivial fibration and as (2) an anodyne map followed by a fibration.*

Proof. In this theorem statement, the terms "fibration" and "trivial fibration" refer to the morphisms of M which become such after applying U while "cofibration" means a map with the left lifting property with respect to all trivial fibrations.

Let I be a set of generating cofibrations and J a set of generating acyclic cofibrations for the model structure on N, both assumed to have small domain[7]. By the correspondence between lifting problems across the adjunction, each element of $F(I)$ has the left lifting property with respect to the trivial fibrations of M, i.e. each element of $F(I)$ is a cofibration. Similarly each element of $F(J)$ is anodyne. Furthermore F preserves small objects (because smallness is about mapping out to directed colimits and U preserves directed colimits) and so $F(I)$, $F(J)$ are sets of morphisms with small domain. Thus the small object argument applies to them and produces a factorization of any map into (1') something in the saturation of $F(I)$ followed by something with the right lifting property with respect to $F(I)$ and (2') something in the saturation of $F(J)$ followed by something with the right lifting property with respect to $F(J)$. Since anodyne maps and cofibrations are defined in terms of lifting properties, the saturations of $F(I)$ and $F(J)$ consist only of cofibrations and of anodyne maps. And finally a map g in M has the right lifting property with respect to $F(I)$ if and only if $U(g)$ has the right lifting property with respect to I if and only if $U(g)$ is a trivial fibration if and only if g is a trivial fibration, and similarly for $F(J)$, meaning (1') and (2') are the desired factorizations. □

Theorem 5.10. *Let N be a model category and M an ordinary category, with an adjunction $F : \mathsf{N} \rightleftarrows \mathsf{M} : U$. The right transferred model structure on M exists if and only if every anodyme morphism of M is a weak equivalence and every morphism in M factors as (1) a cofibration followed by a trivial fibration and (2) an anodyne map followed by a fibration.*

Proof. We omit the "only if" direction. Suppose that anodyne maps are weak equivalences and that factorizations of type (1) and (2) exist for every map. All the model category axioms hold either by passing along the adjunction, by definition of our classes of maps, or by an assumption, as long as the anodyne morphisms and cofibration weak equivalences coincide. Anodyne maps are automatically cofibrations, and we have assumed that they are weak equivalences. Given a cofibration weak equivalence i, factor it as $i = p \circ j$ where j is anodyne and p is a fibration. By the 2-out-of-3 property, p must be a weak equivalence. And since i is a cofibration this implies i has the left lifting property against p. Applying this property to a suitable diagram constructed from the factorization $i = p \circ j$ shows i is a retract of j, and since being anodyne is a lifting property this implies i is anodyne. □

[7] This might be a slightly stronger assumption than cofibrantly generated, I'm not sure.